POU (portition of unity)

$A \subseteq \mathbb{R}^n$. $(\mathcal{U}_i)_{i\in I}$ = loc. finite open cover of A. ($\forall a \in A$ a is cont. in a finite number of $\mathcal{U}_i$)

There exists a collection of smooth fn-ns $\varphi_i : \mathcal{U}_i \to [0,1]$:

1) $\text{supp}\, \varphi_i \subseteq \mathcal{U}_i$ $\quad$ ($\text{supp} f = \overline{\{x : f(x) \neq 0\}}$)

$\uparrow$ compact

$\left(e^{-\frac{1}{(x-a)(x-b)}}\right)$??

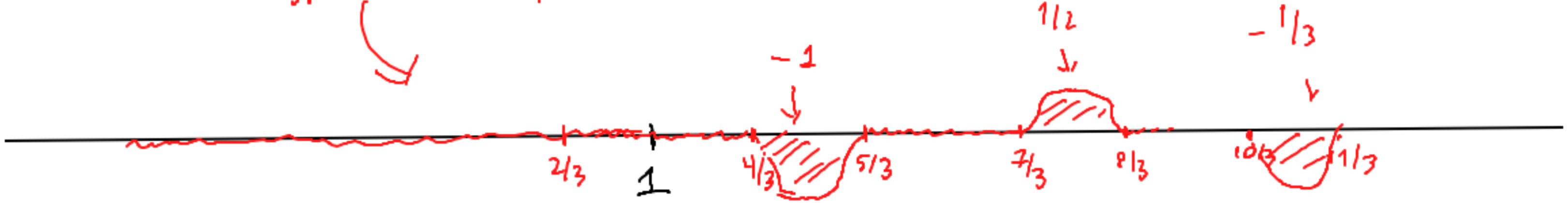
2) $\sum_{i\in I} \varphi_i(a) = 1$

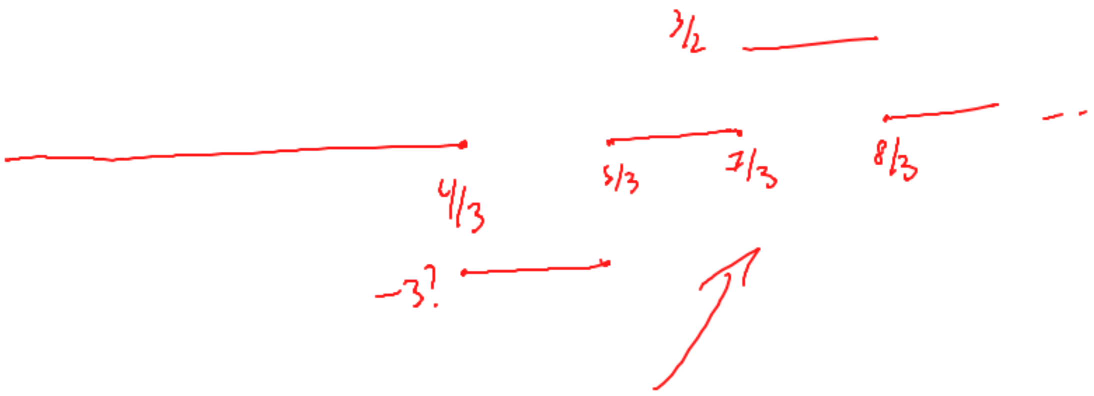
2') this sum is finite $\forall a \in A$.

Q1) $f : \mathbb{R} \to \mathbb{R}$. $\quad f(x) = 0$ for $x < 1$, $\quad \int_{n+1/3}^{n+2/3} f = (-1)^n/n$.

$f|_{(n-1/3,\, n+1/3)} = 0$

smooth example $\quad$ appropriately scaled "bumps"





discontinuous example.

(simpler)

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{-n}^{n} f . \quad \text{(principal value)}$$

$$\int_{-n}^{n} f = \int_{1}^{n} f + \underbrace{\int_{-n}^{1} f}_{\to 0} = \int_{1}^{4/3} + \int_{4/3}^{5/3} + \int_{5/3}^{7/3} + \int_{7/3}^{8/3} + \cdots = \sum_{1}^{n-1} (-1)^n/n.$$

$$0 \qquad -1 \qquad 0 \qquad \tfrac{1}{2}$$

converges conditionally

$$f = f_+ - f_-$$

positive

f is integr $\Leftrightarrow$

f_+, f_- are integr.

$$\int f = \int f_+ - \int f_-$$

(not integrable in this sense)

$$\operatorname{supp} f_+ = \bigcup_n [2n + 1/3,\ 2n + 2/3]$$

$$\operatorname{supp} f_- = \bigcup_n [2n-1 + 1/3,\ 2n-1+2/3]$$

$$\int f_+ = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots = \infty$$

$$\int f_- = -1 - \frac{1}{3} - \frac{1}{5} - \dots = \infty.$$

not integr.

c) "Def" $\int_{\mathbb{R}} f \rightsquigarrow$ $\forall$ partition of unity $(\varphi_i)_{i\in I}$ the sums

$$\sum_{i\in I}\int f\cdot\varphi_i$$

exists and does not depend on the (φ_i).

$$\boxed{1-1/2+1/3-\dots = \ln 2}$$

Thm: If $\sum a_n$ is condit. conv, Then $\forall A\in\mathbb{R}$ $\exists$ permut. $f:\mathbb{N}\to\mathbb{N}$: $\sum a_{f(n)} = A$.

$\sum |a_n| = \infty$.

3-12 $\Rightarrow$ any pou give the same integral if integrable

Def: f is integrable $\Leftrightarrow$ $\displaystyle\sum_{\varphi\in\Phi}\int_{\mathrm{supp}\,\varphi}\varphi\cdot|f| < \infty$.

$$\sum_{\varphi\in\Phi}\underbrace{\int_{\mathrm{supp}\,\varphi}\varphi\cdot|f|}_{1/n} > \int_1^N |f| \quad \forall N > 1.$$

$$\int_1^N |f| = \sum_1^N 1/n = \infty.$$

Q2 f smooth at $a \iff$  $\mathcal{U}$, $\varphi : \mathcal{U} \to \mathbb{R}$: $f|_{A\cap\mathcal{U}} = \varphi|_{A\cap\mathcal{U}}$.

Easy case: $A = \{a\}$.  $V = \mathcal{U}$

If A is any set. $\forall a \in A$ $\exists \mathcal{U}_a \ni a$, $g_a : \mathcal{U}_a \to \mathbb{R}$.

$(\mathcal{U}_a)$ forms an open cover of A.

$\rightsquigarrow$ $(\varphi_a)_{a\in A}$. ← POU.

$$\tilde{f}(a) \stackrel{?}{=} \sum_{a' \in A} \underbrace{g_{a'}(a)\cdot\varphi_{a'}(a)}_{\text{smooth every where}} \stackrel{?}{=} \sum_{a'\in A} f(a)\cdot\varphi_{a'}(a) = f(a) \underbrace{\sum_{a'\in A} \varphi_{a'}(a)}_{= 1.} = f(a)$$

$n=2.$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$

$L(e_1)=e_1$

$L(e_2)=e_1+e_2$

$[-a,a]\times[-b,b] \supseteq L(A)$

$\mu(\mathrm{bnd}(A))=0$

$\Downarrow$

$\mu(\mathrm{bnd}(L(A)))=0$

if $\sum \mathrm{Vol}(U_i) \longrightarrow 0$

$\downarrow$ Lin bdd

$\sum \mathrm{Vol}(L(U_i)) \to 0$

$$\int_{L(A)} 1 \overset{\text{Fubini}}{=} \int_{-b}^{b}\int_{-a}^{a} \chi_{L(A)}(x,y)\,dx\,dy =$$

$\begin{pmatrix} x \\ y \end{pmatrix} \in L(A) \iff L^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \in A$

$$= \int_{-b}^{b}\int_{-a}^{a} \chi_A\left(L^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) dx\,dy =$$

$$= \int_{-b}^{b}\int_{-a}^{a} \chi_A(x-y,y)\,dx\,dy \underset{\substack{\text{1-dim} \\ \text{change}}}{\overset{x \to x+y}{=}} \int_{-b}^{b}\int_{-a-y}^{a-y} \chi_A(x,y)\,dx\,dy =$$

$$= \int_A 1$$

Q1 : $a = \lim_{k\to\infty} a_k$.

b) $a_k = (-1)^n \cdot \frac{n}{n+1}$.

A compact $\iff$ bdd and closed.

$a_k = k$.

$A \cup \{\lim a_k\}$

$\underbrace{a_1 \ldots}_{\text{might be huge!}} a_k, a_{k+1}, \ldots$ lie in ε-nbdd of $\lim a_k$

Lemma: if (a_k) converges, then $\overline{\{a_k\}} \setminus \{a_k\}$ consists of at most one point.

Proof: If $a \neq b$, $a \in \bar{A} \setminus A$, $b \in \bar{A} \setminus A$, then a, b are both limit points

$\Downarrow$

$a = b = \lim a_k$.

c) $\sum_{\phi \in \varphi} \left| \int \phi f \right| < \infty$

$\widetilde{}$

$\sum_{\psi \in \Psi} \left| \int \psi f \right| < \infty$

A rearrangement of series leads to a rearr. of the fn-ns in the partition of unity.

POU from b) won't work!

$\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \ldots$

$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$

$\underbrace{\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{20}}_{1 < \ldots < 2} - 1 + \underbrace{\frac{1}{22} + \frac{1}{24} + \ldots - \frac{1}{3}}_{\frac{1}{2} < \ldots < 1} + \underbrace{\ldots}_{1/4 < \ldots 1/2}$

$b_1 \qquad b_2 \qquad b_3 \ldots$

$\sum b_i = b$

$\#$

$\sum c_i = c$

$\int \psi_1 f = b_1 \qquad \int \psi_2 f = b_2 \quad \ldots$

$\int \phi_1 f = c_1. \qquad \int \phi_2 f = c_2$

(φ_i) $(\mathcal{U}_i)$ cover $\mathbb{R}$

$\varphi_1, \varphi_2, \varphi_3, \varphi_4 \ldots$ pou

$\Downarrow$ exer

If true, prove this!

$A = \mathbb{R}$

$\varphi_1 + \varphi_2, \varphi_3 + \varphi_4, \varphi_5 + \varphi_6 \ldots$ pou?

$\mathrm{supp}(\varphi_1 + \varphi_2) \subseteq \mathrm{supp}(\varphi_1) \cup \mathrm{supp}(\varphi_2) \subseteq \mathcal{U}_1 \cup \mathcal{U}_2.$

$\underbrace{\varphi_1 + \varphi_2 + \ldots + \varphi_{i_k}}, \ldots$