Q3) a) Recall: if $f:[a,b]\to\mathbb{R}$ is cont. $\Rightarrow \forall c: f(a)\le c\le f(b)$

$\exists x: f(x)=c$

Trivial corollary: if $f\in C^1[a,b] \Rightarrow \forall c: f'(a)\le c\le f'(b)$

cont. diff.

$\exists x\in[a,b]: f'(x)=c$

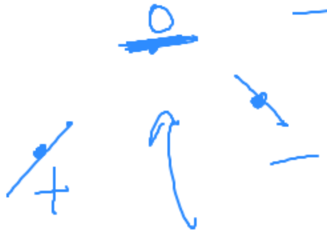
Not-so-triv. corollary: if f is diff. in the interior (a,b), then

$f'(a)\le c\le f'(b) \Rightarrow \exists x\in[a,b]:$

$f'(x)=c$

f' might not be cont.

WLOG:



Use Rolle's thm

if $f'(x)>0$
$f'(y)<0$ $\Rightarrow \exists z: f'(z)=0$

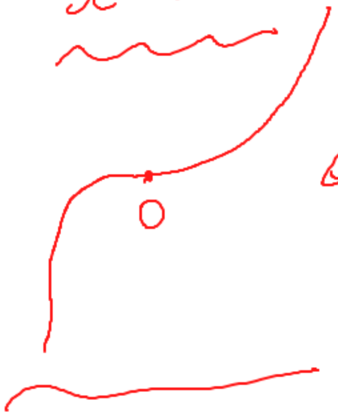
f does not need to be cont. diff.

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \to e^x \begin{pmatrix} \cos y \\ \sin y \end{pmatrix}$$

$$\det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^x \neq 0$$

f misses <u>one</u> value in $\mathbb{R}^2$.

$x \to x^3$



inverse fn-n theorem is not a criterion!

$$\begin{pmatrix} x \\ y \end{pmatrix} \to x^2 + y^2$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to x_1^2 + \dots + x_n^2$$

a general ex where $\det Df = 0$ at $(0, \dots, 0)$ $\longrightarrow$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} x_1^2 + \dots + x_n^2 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{pmatrix}$$

Q1: A is open

$f: A \to \mathbb{R}^n$ is C^1,

$\boxed{f \text{ is } 1\text{-}1.}$ ← necessary

$f'(x)$ is invertible $\forall x \in A$.

Want: $f(A)$ is open

$f^{-1}: f(A) \to A$ is diff'able

$f(B)$ is open $\forall$ open $B \subseteq A$.

(in the ind. top of A)

$\forall x \in A$ $\exists \mathcal{U}_x \ni x$: f is a homeom. on $\mathcal{U}_x$

↑ open nbhd

$f(\mathcal{U}_x)$ is open in $\mathbb{R}^n$

in part. global inverse agrees everywhere with the local inverses diff'able

$A = \bigcup_{x \in A} \mathcal{U}_x$

$f(A) = f\left(\bigcup_{x \in A} \mathcal{U}_x\right) = \bigcup_{x \in A} f(\mathcal{U}_x)$

↑ open

$A \to f(A)$

$f^{-1}(y) \xleftarrow{f^{-1}} y$

locally, f^{-1} is well-def and diff.

$f(x+h) - f(x) = Ah + o(\|h\|)$

being diff'able is a local property!

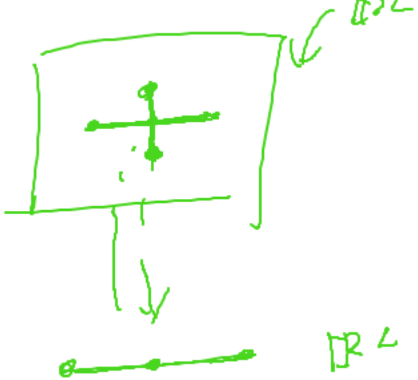
a) I way

Fix $y_0 \in \mathbb{R}$.

$g(x) = f(x, y_0)$.

1) if  g is not 1-1 $\Rightarrow$ done

2) if $g'(x_0) \neq 0$ g is 1-1.



then fix x_0 $f(x_0, y)$

does not generalize!

II way

$$F: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} f(x,y) \\ y \end{pmatrix}$$

assume $\exists x_0, y_0$ $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$.

then F satisfies IFT.

F is locally invertible

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

$a_0 = f(x_0, y_0)$

$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$ has a preim

$\begin{pmatrix} x_0(\varepsilon) \\ y_0(\varepsilon) \end{pmatrix} \to \begin{pmatrix} a_0 \\ b_0 + \varepsilon \end{pmatrix}$ has a preim

The images of $\left[\begin{pmatrix} x_0 - \varepsilon \\ y_0 \end{pmatrix}, \begin{pmatrix} x_0 + \varepsilon \\ y_0 \end{pmatrix}\right]$

$\left[\begin{pmatrix} x_0 \\ y_0 - \varepsilon \end{pmatrix}, \begin{pmatrix} x_0 \\ y_0 + \varepsilon \end{pmatrix}\right]$

have to overlap

$f(x_0(\varepsilon), y_0(\varepsilon)) = a_0$

b) $f : \mathbb{R}^n \to \mathbb{R}^m$ $\quad n > m.$

$$\begin{matrix} x_1 \\ \vdots \\ x_m \\ y_1 \\ \vdots \\ y_{n-m} \end{matrix} \qquad \begin{matrix} x_1 \\ \vdots \\ x_m \end{matrix}$$

$F : \mathbb{R}^n \to \mathbb{R}^n$

$$F\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ y_1 \\ \vdots \\ y_{n-m} \end{pmatrix} = \begin{pmatrix} f_1(x_1 \dots y_{n-m}) \\ \vdots \\ f_m(x_1, \dots y_{n-m}) \\ y_1 \\ y_2 \\ \vdots \\ y_{n-m} \end{pmatrix}$$

$$\underset{\sim}{DF(x,y)} = \begin{array}{c} \left(\begin{array}{c|c} \multicolumn{2}{c}{Df(x,y)} \\ \hline 0 & \begin{matrix} 1 & & \\ & 1 & \\ & & \ddots \; 1 \end{matrix} \end{array}\right) \begin{matrix} m \\ n \end{matrix} \\ n \end{array}$$

$$\underbrace{F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ \hline y \end{pmatrix}}$$

Assume that $\exists (x,y)$:

$\underset{\sim}{\det} DF(x,y) \neq 0.$

$$F\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$\begin{pmatrix} x_0 \\ \hline y_0 \end{pmatrix} \in \mathbb{R}^m \xrightarrow{F}$$

What if

$\underbrace{\det DF(x,y) = 0}$ everywhere?

wrong! $\to$ constant rank thm

$$f(tx) = t^m f(x)$$

$f(x)$ doesn't dep. on t

$$\forall t \in \mathbb{R}$$

dot prod.

$$\frac{d}{dt} f(tx) = Df(tx) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = m t^{m-1} f(x)$$

$t = 1$

$$\underset{\mathbb{R}^n}{x} \to \underset{\mathbb{R}^n}{tx} \to \underset{\mathbb{R}}{f(tx)}$$

$$Df(x) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = m f(x)$$

|| def of part. der-s

$$\sum_i x_i \frac{\partial f}{\partial x_i} = m f(x)$$

$$D_x f(a) = \lim_{t \to 0} \frac{f(a+tx) - f(a)}{t}$$

f is diffable at $a \in \mathbb{R}^n$ $\Rightarrow$ $\forall x \in \mathbb{R}^n$ $f(a+x) - f(a) = Df(a)(x) + o(\|x\|)$

$$f(a+tx) - f(a) = Df(a)(tx) + o(\|tx\|) = t \cdot Df(a)(x) + o(\|tx\|)$$

$$D_x f(a) = \lim_{t \to 0} \frac{f(a+tx) - f(a)}{t} = \lim_{t \to 0} \left(Df(a)(x) + \frac{o(\|tx\|)}{t} \right) = Df(a)(x) + \lim_{t \to 0} \frac{o(\|tx\|)}{t} =$$

$$= Df(a)(x) + \|x\| \lim_{t \to 0} \frac{o(|t|)}{t}$$

0

Let f be C^1. then $\exists\, g_i \in C(\mathbb{R}^n)$:

$f(0)=0$

$$f(x) = \sum_{i=1}^{n} x_i g_i(x)$$

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt$$

$f(x) \in C^1$

$f(0)=0$

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x\,g(x)$

$\frac{f(x)}{x} = g(x)$

$$g = \int_0^1 f'(tx)\,dt =$$