HW4. Q1 familiar!

Example:

1b) $\int_0^x + \int_y^0$

Ex: $F(x) = \int_1^{x^2} f(t)\,dt$

$\frac{d}{dx} F(x)$?

$F : x \xrightarrow[h(x)]{} x^2 \xrightarrow[g(x)]{} \int_1^{x^2} f(t)\,dt$

$h(x) = x^2$

$g(x) = \int_1^x f(t)\,dt$ ← FTC

$F(x) = g(h(x))$

$F'(x) = g'(h(x)) \cdot h'(x) =$

$= (f(h(x)) \cancel{\;\;})h'(x)$

FTC

a) $(x,y) \xrightarrow[\beta]{} x+y \xrightarrow[\alpha]{} \int_0^{x+y} g(t)\,dt$

$f(x,y) = \alpha(\beta(x,y))$

Apply the chain rule in a similar way for all parts

Q2: $\frac{\partial f}{\partial y}(1,y)$

$f(1,y) = \underbrace{1^{2^{3^{xy}}}}_{=1} + \underbrace{\log(1)}_{0}(\ldots\ldots\ldots)$

$f(1,y)=1$

Prob: $\lim_{x\to 1}\frac{\partial f}{\partial y}(x,y) = \frac{\partial f}{\partial y}(1,y)$ ← tricky!

(our f-ns are C^1)

but you don't need this!

I. Let's use the def for $\frac{\partial f}{\partial y}(x_0,y_0)$

$\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} =$

due to the def'n → this is enough! $= \lim_{h\to 0}\frac{f(1,y_0+h)-f(1,y_0)}{h}$

Compl. fact: $\lim_{x_0\to 1}\frac{\partial f}{\partial y}(x_0,y_0) = \frac{\partial f}{\partial y}(1,y_0)$

tricky, verify that every f-n in the expr is C^1 in the nbhd of $(1,y_0)$

Q3 Is an applic. of FTC once again:

$$\frac{\partial}{\partial x}\int_0^x g_1(t,0)\,dt = g_1(x,0)$$

$$\frac{\partial}{\partial y}\int_0^y g_2(x,t)\,dt \overset{FTC}{=} g_2(x,y)$$

(c1) $\frac{\partial f}{\partial x} = x$ $\quad \frac{\partial f}{\partial y} = y$

(c2) $\frac{\partial f}{\partial x} = y$ $\quad \frac{\partial f}{\partial z} = x$

(you can find these fns in the pr. set)

$\frac{\partial f}{\partial x} = 1$ $\quad \frac{\partial f}{\partial y} = 1$

x $\quad y$

$x+y$

$$\frac{\partial f}{\partial x} = g_1$$

$$\Updownarrow$$

$$f(x,y) = \int_0^x \frac{\partial f}{\partial x}(t,y)\,dt = \int_0^x g_1(t,y)\,dt$$

$$\int_0^x g_1(t,y)\,dt = \int_0^y g_2(x,t)\,dt$$

More generally,

mixed der's are same

$$\frac{\partial f}{\partial x} = g_1(x,y) \qquad \boxed{\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}}$$

$$\frac{\partial f}{\partial y} = g_2(x,y) \qquad \Downarrow$$

Q: can we find f?

Also: f exists $\Longrightarrow \frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x}$

$$D_x f(a) = \lim_{t\to 0} \frac{f(a+tx) - f(a)}{t} \qquad e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th position}$$

$$a)\ D_{e_i} f(a) = \lim_{t\to 0} \frac{f(a+te_i) - f(a)}{t} =$$

$$= \lim_{t\to 0} \frac{f(a_1, \dots a_{i-1}, a_i + t, a_{i+1}, \dots a_n) - f(a_1, \dots a_n)}{t} = D_i f(a)$$

$$b)\ D_{\lambda x} f(a) \overset{\text{def}}{=} \lim_{t\to 0} \frac{f(a+t\lambda x) - f(a)}{t} \overset{\substack{\text{rescale} \\ \text{var } t}}{=} \lim_{t\to 0} \frac{f(a + \frac{t}{\lambda}\lambda x) - f(a)}{t/\lambda} =$$

$$= \lim_{t\to 0} \lambda \cdot \dots$$

$$D_{x+y} f(a) = D_x + D_y$$

D is linear!

$$c)\ f(x+h) - f(x) = Df_x(h) + o(\|h\|)$$

$$f(a+tx) - f(a) = \underset{\substack{\| \\ t\,Df_a(x)}}{Df_a(tx)} + \underset{\substack{\| \\ o(t\|x\|) = o(t)}}{o(\|tx\|)}$$

$f: \mathbb{R}^n \to \mathbb{R}$ $\qquad f(tx) = t^m f(x)$

Q6 (Hadamard's lemma)

$$\sum_{i=1}^{n} x_i D_i f(x) = m f(x)$$

Do 1-dimensional case!

$$f(tx) = t^m f(x)$$

$f(0)=0$

First idea: $\frac{d}{dt}$ of both parts

$f(x) = x g(x)$

$$\underbrace{f'(0) = \lim_{x\to 0} \frac{f(x)}{x}} = \underbrace{\lim_{x\to 0} g(x)}$$

$$x \xrightarrow{t} tx \xrightarrow{f} f(tx)$$

$$\left(\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\frac{f(x)}{x} = \frac{f(x)-f(0)}{x-0}$$

$$(x_1, x_2, \ldots x_n) \cdot \begin{pmatrix} \frac{\partial f}{\partial x_1}(tx) \\ \vdots \\ \frac{\partial f}{\partial x_n}(tx) \end{pmatrix} = m \cdot t^{m-1} f(x)$$

$t=1$

$$\sum x_i \cdot \frac{\partial f}{\partial x_i}(x) = m f(x)$$

$$\frac{\partial f}{\partial x_i}(\tilde{x}_1, \tilde{x}_2, \dots \tilde{x}_n) = \lim_{t \to 0} \frac{f(\tilde{x}_1, \dots \tilde{x}_i + t, \dots \tilde{x}_n) - f(\tilde{x}_1, \dots \tilde{x}_n)}{t}$$

$(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)$ diff, direct der-s are discont. at 0

$$f(x+h) - f(x) = \underbrace{Df_x(h)}_{\mathbb{R}^2 \to \mathbb{R}} + o(\|h\|)$$

$f: \mathbb{R} \to \mathbb{R}$

$f(t) \in o(t)$

$\frac{f(t)}{t} \xrightarrow{t \to 0} 0$

$$Df_x = \underbrace{\begin{pmatrix} D_1 f(x) & D_2 f(x) \end{pmatrix}}$$

$D_1 f(x) = D_{e_1} f(x)$, $D_2 f(x) = D_{e_2} f(x)$

$\frac{o(\|h\|)}{\|h\|} \xrightarrow{h \to 0} 0$

$o(h)$ isn't really good because can't d.v. by vector

Some authors force $o(h) \stackrel{\text{def}}{=} o(\|h\|)$

$f(1+h_1, y+h_2) - 1 = D_1 f(1,y) \cdot h_1 + D_2 f(1,y) \cdot h_2 + o(\|h\|)$.

Let's set $h_1 = 0$

$f(1, y+h_2) - 1 = D_2 f(1,y) \cdot h_2 + o(\|h\|)$, where $f(1, y+h_2) = 1$ and $D_2 f(1,y) = 0$

$0 = C \cdot h_2 + o(\|h\|)$ no choice

$C = 0$ due to def'n of $o(\|h\|)$

$g \quad f(tx)$

$$h'(0) = \lim_{t\to 0} \frac{f(tx)-f(0)}{t} = \lim_{t\to 0} \frac{f(tx)}{t} = \lim_{t\to 0} \frac{|t|\,\|x\|\,g\left(tx/\|tx\|\right)}{t} =$$

$$= \lim_{t\to 0} \operatorname{sgn}(t)\,\|x\|\,g\left(tx/|t|\cdot\|x\|\right) = \lim_{t\to 0} \operatorname{sgn}(t)\cdot\|x\|\cdot g\left(\operatorname{sgn}(t)\cdot\frac{x}{\|x\|}\right) =$$

$$= \|x\|\cdot g\left(\frac{x}{\|x\|}\right)$$

$$h'(0) = \lim_{t\to 0}\frac{h(t)-h(0)}{t} = \lim_{t\to 0}\frac{h(t)}{t}$$

$$g(\operatorname{sgn}(t)\cdot\ldots) = \operatorname{sgn}(t)\cdot g(\ldots)$$

So for $h = e_1$, $h = e_2$

$g\left(\frac{e_1}{\|e_1\|}\right) = g(e_1) = 0$

$A(e_1) = 0$

$A(e_2) = 0$

$$\underbrace{f(h_1, h_2) = Ah + o(\|h\|)}$$

$$\frac{d}{dt}h(x) = 0$$

$$f(th_1, th_2) = tAh + o(\|th\|)$$

$$\|h\|\cdot g\left(\frac{h}{\|h\|}\right) = Ah + o(\|h\|)$$

$$\|h\|\cdot g\left(\frac{h}{\|h\|}\right) = \lim_{t\to 0}\frac{f(th)}{t} = Ah + o(\|h\|)$$

$$\lim_{h\to 0} g\left(\frac{h}{\|h\|}\right) = \lim A\left(\frac{h}{\|h\|}\right) = A\left(\lim \frac{h}{\|h\|}\right) \quad 0$$

If f is diffable $\Rightarrow$ $Df(0,0)=0$.

$$f(h) = o(\|h\|)$$

$$\|h\| \cdot g\left(\frac{h}{\|h\|}\right) = o(\|h\|)$$

$$\lim_{h\to 0} g\left(\frac{h}{\|h\|}\right) = 0.$$

$g(e) = 0$ $\forall$ unit vector e.

$$= \frac{|f(x)|}{|x|} \to 0$$

~~there's an assumpt.~~ $\cancel{Df} = 0$.

$\underline{f \in o(|x|)}$

$$\frac{f(x)}{|x|} \to 0$$

$$\left|\frac{f(x)}{|x|}\right| \le |x| \to 0$$

$\Downarrow$

f is diff.

$Df = 0$.