"Diffable" f-ns

Recall: $f:\mathbb{R}^n \to \mathbb{R}^m$ is a "diffable" fn-n at $a \in \mathbb{R}^n$ $\Leftrightarrow$ $\exists$ linear op-r $Df_a(h)$

$$f(a+h) - f(a) = Df_a(h) + \underbrace{o(\|h\|)}_{\text{"small error"}}$$

You want f to be defined

in some open nbhd of a.

Examples: polyn. fn-s are always diff-able.

Recall: for $n=1$ examples of cont, not diff f-ns

$|x|$

weak counterex

$n \geq 2$ Theorem (suff cond for diff): If $\frac{\partial f}{\partial x_i}$ exist $\forall i$ and cont. at a $\Rightarrow$ f is diff at a.

$$\frac{\partial f(x_1, \ldots x_i, \ldots x_n)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots x_i + h, \ldots x_n) - f(x_i)}{h} = \frac{\partial f}{\partial x_i}$$

part.

$\frac{\partial f}{\partial x} \leftarrow$ number

partial deriv.

Theorem: $f: \mathbb{R}^n \to \mathbb{R}$. Suppose $\forall\, 1 \le i \le n$ $\frac{\partial f}{\partial x_i}$ exist and are cont. at $a \in \mathbb{R}^n$.

Then f is diffable at a. $Df_a = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$

$g := x^2 \sin\left(\frac{1}{x}\right)$ is weird: g is cont. everywhere

g is diff. everywhere (come up with an arg. for $x=0$)

g' is discont at $x=0$.

useful counterex.

$$\boxed{\begin{aligned} \tilde{g}(0,0) &= 0. \\ \tilde{g}(x,y) &= (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \quad x^2+y^2 \neq 0 \end{aligned}}$$

Exer: 1) $\tilde{g}$ is cont., but $\frac{\partial g}{\partial x}$ $\frac{\partial g}{\partial y}$ are discont. at 0 (resp. limits do not exist)

2) $\tilde{g}$ is diffable at 0.

Example:

$C^3 x^2$

$f(x,y) = x^2 y^3$.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = \frac{\partial (x^2 y^3)}{\partial x} = 2xy^3$$

("you fix all other variables")

$$\frac{\partial f}{\partial y} = 3x^2 y^2.$$

f is diff-able at $a \Rightarrow$ f is cont. at a.

$$\frac{o(f)}{f} \to 0$$

Q1 immed from def-n:

you did this in 1-dim case

$$\lim_{h \to 0} \left(f(a+h) - f(a)\right) = \lim_{h \to 0} \left(Df_a(h) + o(|h|)\right)$$

linear $\Rightarrow$ cont.

$$\lim Df_a(h) = Df_a(\lim h)$$

Df

Q3 $f: \mathbb{R}^n \to \mathbb{R}$, $|f(x)| \leq |x|^2$

Want to: Find ~~[illegible]~~ such linear functional L: $f(x) = \underline{L(x)} + o(|x|)$

if x is close to 0

x^2 is extremely small

Hint: $|x|^2 \in o(|x|)$

use the def-n of $o(-)$

Warning:

naive gen from $\mathbb{R}^1$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

wrong! (can't div. by a vector)

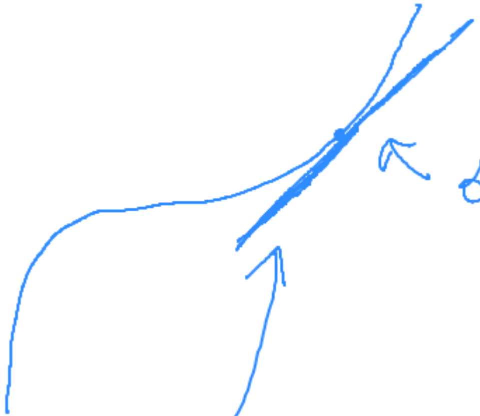
$f: \mathbb{R}^n \to \mathbb{R}$

Correct:

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Lh}{\|h\|} = 0$$

the same as writing $o(\|h\|)$

An exception to this is a complex deriv.



derivative

linear

approx of f at x

tangent line

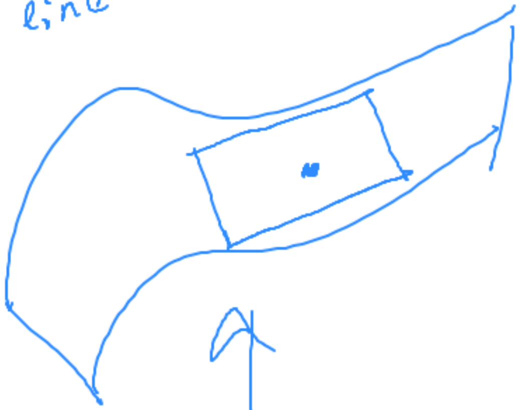
is the

1-dim. version

of the tangent

space

of a function



notion of

tangent ~~[illegible]~~ space

$$f(x+h) - f(x) = \underset{\sim}{k} \cdot h + o(h)$$

10^{-12}

$h \mapsto k \cdot h$

$\mathbb{R} \to \mathbb{R}$

$\left(\frac{1}{2600}\right)^2$

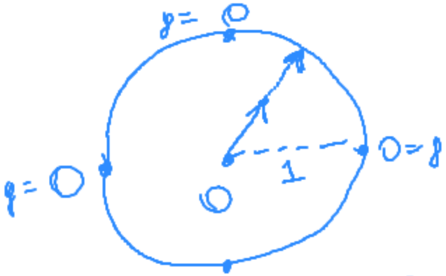
$$f(x+h) - f(x) = L h + o(\|h\|)$$

$O(\|h\|^2)$

error

linear approx to error

extremely small



$S^1 \subseteq \mathbb{R}^2$

$g : S^2 \to \mathbb{R}$. given

1) g is cont

2) $g(0,1) = g(1,0) = 0$

3) $g(-x) = -g(x)$

odd

S^2

$g \Rightarrow f : \mathbb{R}^2 \to \mathbb{R}$

$f(x) = \|x\| \cdot g\left(\frac{x}{\|x\|}\right)$.

Keep in mind that

$\left\|\frac{x}{\|x\|}\right\| = 1$.

$\Uparrow$

S^1

$f(0) = 0$

a) $\forall x \neq 0 \in \mathbb{R}^2$

$f_x(t) = f(xt)$

prove that f is diff at 0.

Hint: expand in terms of g,

use the def-n of der.

b) Try comp. $f_x'(t)$ for diff x.

Q4: U is open

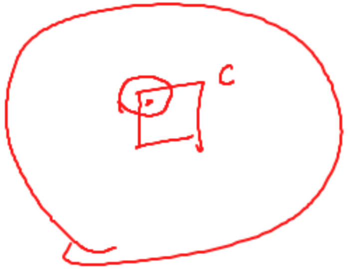
compact $C \subset \mathcal{U}$

$X_0 \subseteq X$

$\overline{X_0}$ = closure

$X_0 \cup \text{Bnd}(X_0)$, ~~[illegible]~~ closed superset.

Let's assume $\mathcal{U}$ is bounded



V-open "ε"-nbhd of C in $\mathcal{U}$

$\forall x \in C$

$B_x \subsetneq \mathcal{U}$ $\quad \overline{B_x} \subsetneq \mathcal{U}$

$\overline{V}$ Fact: inf unions of comp of one not comp.

You can use chain rule

$e^{y \ln x}$ composition of

$(x,y,z) \to (x,y)$

$\mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R} \to \mathbb{R}$

$x \to e^x$

$(a,b) \to (\ln a) \cdot b$

less coordinates

("coord-free approach")

Q6: Related problem:

A is a 2×2 real matrix

$f_A(t) = \det(E + tA)$. (E identity)

$f_A'(0) = tr(A)$.

Hint: think about char. poly of A.

$$f(x,y) = xy$$

$K \in \mathbb{R} \rightsquigarrow x \to Kx$

not obv! $\mathbb{R}^2$

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{\|h\|} = \lim_{h\to 0} \frac{(x+h_1)(y+h_2) - xy - L(h)}{\sqrt{h_1^2+h_2^2}} = \lim_{h\to 0} \frac{xh_2 + yh_1 + h_1h_2 - L(h)}{\sqrt{h_1^2+h_2^2}}$$

$h = (h_1, h_2)$ define

$a = (x, y)$ $L(h) = \langle h \mid \binom{y}{x} \rangle$

deg 2

$$\lim_{h\to 0} \frac{h_1h_2}{\sqrt{h_1^2+h_2^2}} = 0$$

$f : x \longmapsto 2x+3$

$f'(x) = 2$

compon. of differ.

$$\lim_{h\to 0} \frac{xh_2 + yh_1}{\sqrt{h_1^2+h_2^2}}$$

first comp. is y

second is x

deg 1

$L(h)$ $o(\|h\|)$

$$\lim_{h\to 0} \frac{\langle \binom{h_1}{h_2} \mid \binom{y}{x} \rangle + h_1h_2}{\sqrt{h_1^2+h_2^2}}$$

$$\frac{\partial f}{\partial x} = \frac{\partial (xy)}{\partial x} = y$$

numbers

$$\frac{\partial f}{\partial y} = \frac{\partial (xy)}{\partial y} = x$$

numbers

~~Open~~ ~~+~~ bounded + closed $\Longrightarrow$ compact

$\Longleftarrow$

K is comp.

$\mathbb{R}^n \setminus K$ is open

Consider: $\mathcal{U}_r = \mathbb{R}^n \setminus \overline{B_x(r)}$

$\mathcal{U}_r$ is open

$\mathcal{U}_r$ covers K.

$\mathcal{U}_r \longrightarrow \mathbb{R}^n \setminus x$

$r \to 0$

$r_1 < r_2$

$\mathcal{U}_{r_1} \supset \mathcal{U}_{r_2}$

$\{\mathcal{U}_r\}$ cover K.

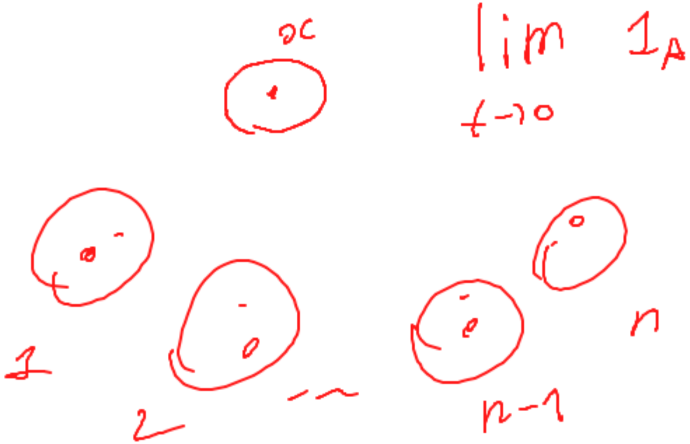
$\mathcal{U}_{r_1}, \ldots, \mathcal{U}_{r_n}$ cover K

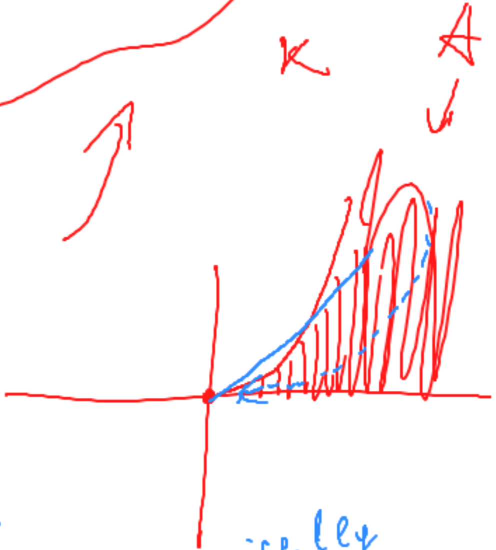
$\{U_y\}_{y\in K}$ — cover of K

$$\lim_{(x,0)} 1_A(x) = 0$$

$$\lim_{t\to 0} 1_A(\gamma(t)) = 1$$

curve inside A, conv. to 0





1_A isn't cont in $\mathbb{R}^2$ because the preim. of 1_A doesn't behave nicely

$1_A : \mathbb{R}^2 \to \mathbb{R}$

$x \in A \to 1$

$x \notin A \to 0$

~~$\lim_{x\to 0} 1_A(x) = ?$~~

~~$\lim_{(x,0)} 1_A(x) = ?$~~

~~$[0,x]$~~

1_A is not cts at 0

$1_A(tx)$ is cts at 0 $\forall x \in \mathbb{R}^2$