CS inequality

Inner prod. spaces

Normed space ?

V - vector space $/\mathbb{R}$

$\langle \cdot | \cdot \rangle$:

strong!

1) $\langle a|a\rangle = 0 \iff a = 0$ non degen.
$\langle a|a\rangle \geq 0$

2) $\langle \lambda v + \beta w | u\rangle = \lambda \langle v|u\rangle + \beta \langle w, u\rangle$ bilinear

3) $\langle v, w\rangle = \langle w, v\rangle$ symmetric

$\forall x \in V$

$\|x\| = \sqrt{\langle x|x\rangle}$

$|\langle x|y\rangle| \leq \|x\| \cdot \|y\|$

Def: A function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is called a norm if

1) $\|x\| \geq 0$, $\|x\| = 0 \iff x = 0$

2) $\|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall \lambda \in \mathbb{R}, x \in V$

3) $\|x+y\| \leq \|x\| + \|y\|$ (triangle ineq.)

open $\Rightarrow$ compl. is open $\quad X/A = A^c$

closed $\Rightarrow$ complem. is open $\quad (A^c)^c = X/(X/A) = A$

Prop: $\sqrt{\langle x|x\rangle} := \|x\|$ is a norm

$\langle x+y | x+y\rangle$

Inner product $\Longrightarrow$ norm

?

Inner product $\Longleftarrow$ norm

Prop: Not every normed space is an inner prod. space!!!

some authors use this to den. norm

$|\cdot|$ - abs. v.

$\|\cdot\|$ - norm

Parallelog. law:

In an inner prod. space

squared!

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$\|x+y\|^2 = \langle x+y | x+y \rangle$

$\langle x|y \rangle$

$-\langle x|y \rangle$

Exer: show that $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ are norms!!!

Examples:

Let $V = \mathbb{R}^n$.

For $p \in [1, \infty]$ define, $x \in \mathbb{R}^n$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p\right)^{1/p}$$

(p-norm)

(sup-norm) $\|x\|_\infty = \max_{\sup} \{|x_1|, \dots |x_n|\}$. ?

P1 $\|\cdot\|_1$ is a norm

$p=1$ $\quad |x_1| + \dots + |x_n|$

$p=2$ $\quad \sqrt{|x_1|^2 + \dots + |x_n|^2}$ Euclidian norm

$p=\infty$

Prop: $\|\cdot\|_p$ is a norm for all $p \in [1, \infty]$,

(not entirely trivial!)

The triangle is not trivial

in

Hint: 1) Prove the Young's ineq.: $\forall p, q$ $\boxed{\frac{1}{p} + \frac{1}{q} = 1}$ $a, b \in \mathbb{R}$

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$$

2) Prove the Hölder's ineq: $x, y \in \mathbb{R}^n$

$$\langle x | y \rangle \leq \|x\|_p \|y\|_q$$

$\frac{1}{2} + \frac{1}{2} = 1$

for $p = q = 2$

this CS

~~Pf~~

Prop: Denote $(\mathbb{R}^n, \|\cdot\|_1) = \ell^1_{(n)}$ ← "el-one space"

Then ℓ^1 is not an inner prod. space.

(e.g. no $\langle x|x\rangle = \|x\|_1$)

$$\|v+w\|_1^2 + \|v-w\|_1^2 = 2\left(\|v\|_1^2 + \|w\|_1^2\right)$$

Proof: P. law

$n=2$ $\quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

Exercise: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

Counterexample: $\left\|\begin{pmatrix} -1 \\ -2 \end{pmatrix}\right\|_1^2 + \left\|\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\|_1^2 = 2(1+16) = 34$

$9 \qquad 25$ not a counter-ex.!

$|3|+|2| = 5$

Prop: $(\mathbb{R}^n, \|\cdot\|_p) = \ell^p_{(n)}$. $\ell^p_{(n)}$ is an inner prod. space $\iff p=2$.

Q.4 $(\mathbb{R}^n)^*$ - dual space of $\mathbb{R}^n$ $(\mathbb{R}^n)^* = \{\mathbb{R}^n \to \mathbb{R}\}$ linear

Idea: $(\mathbb{R}^n)^*$ is canonically isomorphic to $\mathbb{R}^n$

(not a proof: $\dim \mathbb{R}^n = \dim (\mathbb{R}^n)^*$)

$y \mapsto \langle x | y \rangle$ can not be id zero! for $x \neq 0$

$\langle x | x \rangle \neq 0$

$\mathbb{R}^n$ is inner prod. space

Riesz (?) rep:

Prop: φ is an isomorphism.

$\varphi : \mathbb{R}^n \to (\mathbb{R}^n)^*$

Proof: I. φ is injective follows $\langle \cdot | \cdot \rangle$ is non-deg.

$x \to \langle x | \cdot \rangle$

$(y \mapsto \langle x | y \rangle)$

If f is a linear functional (use orthogonality: $\mathbb{R}^n = \ker f \oplus \langle v \rangle$)

works for ~~any~~ inner prod not just the stand one)

II. φ is surj.

$\forall x \in \mathbb{R}^n \quad x = \lambda_1 e_1 + \dots + \lambda_n e_n$

$\lambda_1 \langle e_1 | \cdot \rangle + \dots + \lambda_n \langle e_n | \cdot \rangle$

Same holds for any $\underbrace{\text{Hilbert space}}$

is a complete (w.r.t. to the $\leftarrow \|\cdot\| = \langle\cdot|\cdot\rangle$)
inner prod.
space

$$\ell^2 = \left\{ (a_1, a_2, a_3, \dots) \;\middle|\; \sum_{i=1}^{\infty} (a_i)^2 < \infty \right\}$$ $\leftarrow$ Huge!!! space

$$\langle a | b \rangle = \sum_{i=1}^{\infty} a_i b_i$$

Prop: $(\ell^2, \langle\cdot|\cdot\rangle)$ is an i.p.s.
it is complete w.r.t. $\|\cdot\|$

Any cont. linear funct. on ℓ^2 will be repr. like $\langle a|\cdot\rangle$.

$T : \mathbb{R}^n \to (\mathbb{R}^n)^*$

$x \to \varphi_x$

$\varphi_x(y) = \langle x | y \rangle$

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6. $\mathbb{Q} \subset A$

$\forall x \in [0,1] \setminus \mathbb{Q}$.

x lies in the bound.

~~Any~~ open nbhd of x intersects A and $[0,1] \setminus A$

1) Any nbhd inter with A

2) Any nbhd inter. $[0,1] \setminus A$

Use: open nbhd = open interval

(open interval has one ~~rat~~ rational one irrat)

Need to prove that

Q6: $(\underbrace{\mathrm{Int}(A)}_{A} \cup \mathrm{Ext}(A))^c = \emptyset$

$A \cup \mathrm{Ext}(A)$

T is inj and surj.

1) T is inj.

$x \neq 0 \Rightarrow \varphi_x \neq 0$. non-n ~~egos~~.

$\varphi_x(x) = ?$

2) T surj:

I. coordinates

$\langle e_i | \cdot \rangle$

~~II~~. Orthog. inner prod structure

compl. to

$\ker \varphi_x$

$(\mathbb{R}^{\mathbb{N}})^* \not\cong \mathbb{R}^{\mathbb{N}}$

(coordin. issues)

$\mathbb{R}^{\mathbb{N}}$ countb. basis

$(\mathbb{R}^{\mathbb{N}})^*$ uncountb. basis

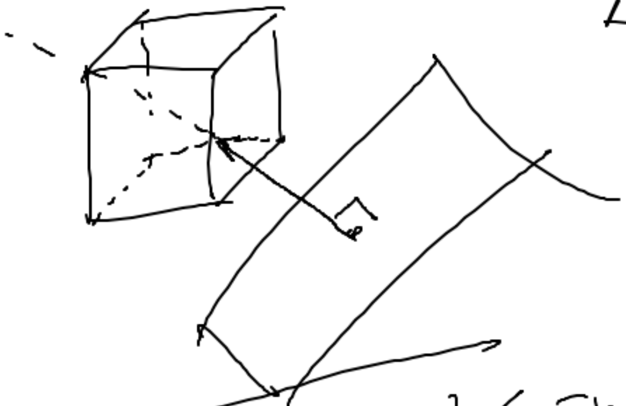
$T: \mathbb{R}^m \to \mathbb{R}^n$

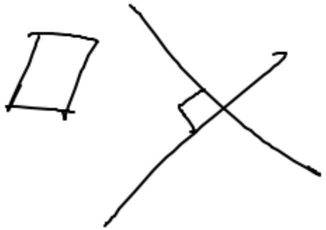
$\|Th\|_2 \leq M \|h\|_2$.

$h = (h_1, \dots h_n)$

$$\left(\sum_{i=1}^{n} h_i\right)^2 = \langle h \mid 1 \rangle^2 \leq \|h\|^2 \cdot \|1\|^2$$

$1 = (1, 1, \dots 1)$, $\quad \|1\|^2 = n$





$$\left(\sum h_i\right)^2 \leq n \|h\|^2$$

$\sqrt{a_1^2 + \dots + a_n^2} \leq \sum |a_i|$

Q1:

$\|\cdot\|_2 \leq \|\cdot\|_1$, $\leftarrow$ Hölder for $(1, \infty)$

$\langle x | y \rangle \leq \|x\|_p \|y\|_q$ $\quad p = 1$, $q = \infty$

Try to prove: $1 \leq p < q \leq \infty \Rightarrow \quad \|\cdot\|_q \leq \|\cdot\|_p$

$a_1^2 + \ldots + a_n^2 \; ? \; (a_1 + \ldots + a_n)^2$