MAT257 Term Test 2 Rejects

The following questions were a part of a question pool for the 2020-21 MAT257 Term Test 2, but at the end, they were not included.

1. Let A be a rectangle in $\mathbb{R}^n$ and let $f, g\colon A \to \mathbb{R}$, where f is integrable on A and g is equal to f except on finitely many points. Show from basic definitions that g is also integrable on A and that $\int_A f = \int_A g$.

   Tip. "From basic definitions" means "not using any of the theorems that came after the definitions that are necessary to make the question meaningful". In our case those definitions are those of lower and upper sums, integrability, and the integral. Yet words like "measure-0", whether or not they are relevant, are forbidden.

2. (a) Show that the boundary of a set of content-0 is also of content-0.
   (b) Give an example of a set of measure-0 whose boundary is not of measure-0.

3. Show that if $f\colon A \to \mathbb{R}$ is integrable on a rectangle A, and if $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and bounded on $f(A)$, then $g \circ f$ is also integrable on A.

4. Show that if a set $A \subset \mathbb{R}^n$ is Jordan measurable, then there is a finite collection $\mathcal{R}$ of nearly disjoint rectangles in $\mathbb{R}^n$ (meaning, disjoint except perhaps for their boundaries) and a sub-collection $\mathcal{R}'$ of $\mathcal{R}$, such that

$$(1) \qquad \bigcup_{R\in\mathcal{R}'} R \subset A \subset \bigcup_{R\in\mathcal{R}} R \qquad \text{and} \qquad \sum_{R\in\mathcal{R}\setminus\mathcal{R}'} v(R) < \frac{1}{257}.$$

5. (a) Show that if a set B is bounded, has measure 0, and its characteristic function χ_B is integrable, then $\int \chi_B = 0$.
   (b) Give an example of a bounded measure-0 set whose characteristic function is not integrable.

6. Show that if a non-negative continuous function f defined on some rectangle A in $\mathbb{R}^n$ has integral equal to 0, then f is identically equal to 0.

7. Prove Young's inequality: if f is a continuous strictly increasing function on $\mathbb{R}$ with $f(0) = 0$ and if a and b are non-negative numbers, then

$$\int_0^a f(x)dx + \int_0^b f^{-1}(y)dy \geq ab.$$

   Hint. Draw the graph of f and try to interpret the two integrals and the product ab as areas.

8. Suppose A and B are two Jordan measurable subsets of $\mathbb{R}^n$ that have the property that for every $t \in \mathbb{R}$, the $(n-1)$-dimensional volume of the slice of A at height t (meaning, of $\{x \in \mathbb{R}^{n-1}\colon (x,t) \in A\}$) is equal to the $(n-1)$-dimensional volume of the slice of B at height t. Prove that $v(A) = v(B)$.

9. Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth and has an invertible differential at 0, then near 0 it can be written as a composition $T_n \circ g_n \circ \cdots \circ T_2 \circ g_2 \circ T_1 \circ g_1 \circ T_0$, where each T_i is a "permutation map" that merely permutes the coordinates of $x = (x_1\ x_2\ \ldots\ x_n) \in \mathbb{R}^n$, and each g_i changes the value of only the last coordinate; precisely, $g_i(x_1\ x_2\ \ldots\ x_n) = (x_1\ \ldots\ x_{n-1}\ h_i)$, where the $h_i\colon \mathbb{R}^n \to \mathbb{R}$ are smooth.

10. Let f be a possibly-unbounded function defined on a possibly-unbounded open subset A of $\mathbb{R}^n$, and assume that f is integrable on A. Let B be an open subset of A. Prove that f is integrable also on B.

11. Compute the volume of the ellipsoid $\{(x, y, z)\colon 2x^2 + 3y^2 + 5z^2 \leq 1\}$. You may use the fact that the volume of the ball of radius R in $\mathbb{R}^3$ is $\frac{4}{3}\pi R^3$.

12. Let $f\colon \mathbb{R}^2_{x,y} \to \mathbb{R}$ be a function that has continuous second derivatives, and let A be a rectangle in $\mathbb{R}^2_{x,y}$

    (a) Use Fubini to show that $\int_R \partial_x(\partial_y f) = \int_R \partial_y(\partial_x f)$.

    (b) Use the above result to show that $\partial_x(\partial_y f) = \partial_y(\partial_x f)$.

13. Let T_θ be the "rotation by θ" matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and let B be a Jordan-measurable subset of $\mathbb{R}^2$. Prove that $v(B) = v(T_\theta B)$.

14. Let A be a subset of $\mathbb{R}^n$, and let $\mathcal{U}$ and $\mathcal{V}$ be open covers of A. Show that if $\{\phi_i\}$ is a partition of unity for A subordinate to $\mathcal{U}$ and $\{\psi_j\}$ is a partition of unity for A subordinate to $\mathcal{V}$, then $\{\phi_i\psi_j\}$ is a partition of unity for A subordinate to $\mathcal{W} := \{U \cap V\colon U \in \mathcal{U},\ V \in \mathcal{V}\}$.

15. Show that if $\mathcal{U}$ is an open cover of some set $A \subset \mathbb{R}^n$, then $\mathcal{U}$ has a countable subcover. *Hint.* It helps to know that there are countably many rectangles in $\mathbb{R}^n$ whose corners all have rational coordinates, and that every open set is a union of such rectangles.

16. Let a_i for $i = 1, \ldots, p$ be a finite collection of distinct points in $\mathbb{R}^n$ and let b_i, for $i = 1, \ldots, p$ be real numbers. Prove that there exists a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(a_i) = b_i$.

17. Let $f\colon [a, b] \to \mathbb{R}$ be a non-decreasing function and let $\epsilon > 0$. Directly from the definitions, show that there is a partition P of $[a, b]$ such that $U(f, P) - L(f, p) < \epsilon$ (and hence f is integrable).