$\boxed{G1}$

$\boxed{\text{lemma}}$ I claim if T & $F \in B$

& T & F agree on all vectors $\{ (v_i, w_j) : \begin{matrix} 1 \le i \le n \\ 1 \le j \le m \end{matrix} \}$

then $T = F$.

$\boxed{\text{pf}}$

$$T(v, w) = T\left(\sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{m} \beta_j w_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \, T(v_i, w_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \, F(v_i, w_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \, F(v_i, w_j)$$

$$= F(v, w)$$

This is true for all $v, w \in V$ & W respectively

& so $T = F$.

$\square$

Now define $T_{ij} \in B$ such that

$$T_{ij}(v_l, w_k) = \begin{cases} 1 & (i,j) = (l,k) \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \begin{array}{l} 1 \le l \le n \\ 1 \le k \le m \end{array}$$

for all $(i,j) \in \{1, \dots, n\} \times \{1, \dots, m\}$.

This is true for all $v, w \in V$ & W respectively

& so $T = F$

$\square$

Now define $T_{ij} \in B$ such that

$$T_{ij}(v_\ell, w_k) = \begin{cases} 1 & (i,j) = (\ell, k) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \begin{array}{l} 1 \le \ell \le n \\ 1 \le k \le m \end{array}$$

for all $(i,j) \in \{1, \dots, n\} \times \{1, \dots, m\}$.

I claim $\{T_{ij} : 1 \le i \le n,\ 1 \le j \le m\}$ is a basis for B

$T_{ij} := \phi_i \otimes \psi_j$ has this property

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linearly independence:

Assume $\sum_{\substack{1\le i\le n \\ 1\le j\le m}} \alpha_{ij} T_{ij} = 0$ Apply this to the vector (v_l, w_k)

Then $\alpha_{lk} = 0$

This is true for all $1\le l\le n$, $1\le k\le m$

4 hence linearly independent.

this is true for all $1 \le \ell \le n$, $1 \le k \le m$

& hence linearly independent.

<u>Spans</u>

Let $F \in B$. Then let $\delta_{ij} := F(v_i, w_j)$

Then I claim $F = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} \delta_{ij} T_{ij}$

To see this, notice F & $\sum \delta_{ij} T_{ij}$ agree

on all vectors $\{(v_i, w_j) : 1 \le i \le n, 1 \le j \le m\}$

& so by lemma are equal.

$1 \le j \le m$

To see this, notice F & $\sum \delta_{ij} T_{ij}$ agree

on all vertices $\{(v_i, w_j) : 1 \le i \le n, 1 \le j \le m\}$

& so by lemma are equal.

Hence this forms a basis for B.

There are nm different T_{ij} & so the dimension is nm.

$\square$

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$\boxed{62}$ $\omega = xy\,dx + 2x\,dy - y\,dz \in \Omega^1(\mathbb{R}^3_{x,y,z})$

$$\alpha : \mathbb{R}^2_{u,v} \longrightarrow \mathbb{R}^3_{x,y,z} \qquad \alpha(u,v) = \begin{pmatrix} uv \\ u^2 \\ 3u+v \end{pmatrix}$$

$$D\alpha(u,v) = \begin{bmatrix} v & u \\ 2u & 0 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2u & 0 \\ 3 & 1 \end{bmatrix}$$

- $dw = d(xy\,dx + 2x\,dy - y\,dz)$

$= d(xy\,dx) + 2\,d(x\,dy) - d(y\,dz)$

$= d(f_1) \wedge dx + 2\,d(f_2) \wedge dy - d(f_3) \wedge dz$

where $f_1(x,y,z) = xy \quad f_2(x,y,z) = x \quad f_3(x,y,z) = y$

$= (D_2 f_1\, dy + D_3 f_1\, dz) \wedge dx + 2(D_1 f_2\, dx + D_3 f_2\, dz) \wedge dy$

$- (D_1 f_3\, dx + D_2 f_3\, dy) \wedge dz$

$$= d(f_1) \wedge dx + 2\, d(f_2) \wedge dy - d(f_3) \wedge dz$$

where $f_1(x,y,z) = xy \quad f_2(x,y,z) = x \quad f_3(x,y,z) = y$

$$= (D_2 f_1\, dy + D_3 f_1\, dz) \wedge dx + 2(D_1 f_2\, dx + D_3 f_2\, dz) \wedge dy$$

$$- (D_1 f_3\, dx + D_2 f_3\, dy) \wedge dz$$

$$= x\,dy \wedge dx + 2\, dx \wedge dy - dy \wedge dz$$

$$= -x\, dx \wedge dy + 2\, dx \wedge dy - dy \wedge dz$$

$$= (2-x)\, dx \wedge dy - dy \wedge dz$$

$$\alpha^* \omega = \alpha^* (xy\,dx + 2x\,dy - y\,dz)$$

$$= \alpha^*(xy) \wedge \alpha^* dx + 2\alpha^*(x) \wedge \alpha^* dy - \alpha^*(y) \wedge \alpha^* dz$$

ASIDE: Now $\alpha^*(xy)(u,v) = \alpha^*(f_1)(u,v)$

$$= (f_1 \circ \alpha)(u,v)$$

Note: f_1, f_2, f_3 defined on

$$= f_1(uv, u^2, 3u+v)$$

$$= u^3 v$$

$\alpha^*(xy) \wedge \alpha^* dx + 2\alpha^*(x) \wedge \alpha^* dy$

ASIDE: Now $\alpha^*(xy)(u,v) = \alpha^*(f_1)(u,v)$

$$= (f_1 \circ \alpha)(u,v)$$

Note: f_1, f_2, f_3 defined on last page.

$$= f_1(uv, u^2, 3u+v)$$

$$= u^3 v$$

$$\alpha^*(x)(u,v) = \alpha^*(f_2)(u,v)$$

$$= (f_2 \circ \alpha)(u,v)$$

$$= f_2(uv, u^2, 3u+v)$$

$$= uv$$

$$\alpha^*(y)(u,v) = \alpha^*(f_3)(u,v)$$
$$= (f_3 \circ \alpha)(u,v)$$
$$= f_3(uv, u^2, 3u+v)$$
$$= u^2.$$

Also: $\alpha^* dx = d\alpha_1$
$$= v\,du + u\,dv$$

$$\alpha^* dy = d\alpha_2$$
$$= 2u\,du$$

$$\alpha^* dz = d\alpha_3$$
$$= 3du + dv.$$

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Then $(\alpha^* \omega)(u,v) = u^3 v\,(v\,du + u\,dv)$

$+\ 2uv\,(2u\,du)$

$-\ u^2\,(3du + dv)$

$= u^3v^2\,du + u^4 v\,dv$

$+\ 4u^2 v\,du$

$-\ 3u^2 du - u^2 dv$

$$- 3u^2 du - u^2 dv$$

$$= (u^3v^2 + 4u^2v - 3u^2) du$$

$$+ (u^4v - u^2) dv.$$

$$d(\alpha^* \omega) = d\left((u^3v^2 + 4u^2v - 3u^2) du + (u^4v - u^2) dv \right)$$

$$= d(u^3v^2 + 4u^2v - 3u^2) \wedge du$$

$$+ d(u^4v - u^2) \wedge dv$$

$$= (2u^3v + 4u^2 -) \, dv \wedge du$$

$$= d(u^3v^2 + 4u^2v - 3u^2) \wedge du$$

$$+ d(u^4v - u^2) \wedge dv$$

$$= (2u^3v + 4u^2) \, dv \wedge du$$

$$+ (4u^3v - 2u) \, du \wedge dv$$

$$= \left[(-2u^3v - 4u^2) + (4u^3v - 2u) \right] du \wedge dv$$

$$= (+2u^3v - 2u - 4u^2) \, du \wedge dv$$

• $\alpha^*(d\omega) = d(\alpha^*\omega)$ = see last page

To make sure:

$$\alpha^*\big((2-x)\,dx\wedge dy - dy\wedge dz\big)$$

$$= \alpha^*(2-x)\,\alpha^*(dx\wedge dy) - \alpha^*(dy\wedge dz)$$

$$= (2-uv)\det\begin{pmatrix} v & u \\ 2u & 0 \end{pmatrix} du\wedge dv - \det\begin{pmatrix} 2u & 0 \\ 3 & 1 \end{pmatrix} du\wedge dv$$

$(\alpha^*(2-x))(u,v)$

$= (2-x)(\alpha(u,v))$

$= (2-x)(uv, u^2, 3u+v)$

$= 2-uv$

$$= (2-uv)(-2u^2)\,du\wedge dv - 2u\,du\wedge dv$$

$$= (-4u^2 + 2u^3v - 2u)\,du\wedge dv$$

!! ☺

Hence $dc = c_1 - c_0$. $\square$

$\boxed{6.3}$

Let V denote the set of vector fields on $\mathbb{R}^3$. (that are smooth)

Then in diagrams:

$$\begin{tikzcd} V \arrow[r, "\mathrm{curl}"'] \arrow[d, "\alpha"'] & V \arrow[d, leftrightarrow, "\beta"] \\ \Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \end{tikzcd}$$

where α & β are vector space isomorphisms as defined

as follows:

$$\Omega^1(\mathbb{R}^3) \xrightarrow[d]{} \Omega^2(\mathbb{R}^3)$$

where α & β are vector space isomorphisms as defined as follows:

if $F \in V$, then $F(x) = (x; \sum_{i=1}^{3} f_i(x) e_i)$

$$\alpha(F) := \sum_{i=1}^{3} f_i \, dx_i$$

& $$\beta(F) := \sum_{i=1}^{3} f_i \, \underbrace{dx_{no\ i}} \, (-1)^{i-1}$$

represents $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_3$

<u>Step 1:</u> Check that α & β are isomorphisms.

$$\alpha(F) := \sum_{i=1}^{3} f_i \, dx_i$$

& $$\beta(F) := \sum_{i=1}^{3} f_i \underbrace{dx_{\text{no } i}} (-1)^{i-1}$$

↳ represents $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge d$

Step 1: Check that α & β are isomorphisms.

Let $F(x) = \left(x, \sum_{i=1}^{3} f_i(x) e_i\right)$

& $G(x) = \left(x; \sum_{i=1}^{3} g_i(x) e_i\right)$ where $F, G \in V$

Surjective

i) Claim: α is an isomorphism.

linear: $\alpha(\lambda F + G) = \sum_{i=1}^{3} (\lambda f_i + g_i) e_i$

$= \lambda \sum_{i=1}^{3} f_i e_i + \sum_{i=1}^{3} g_i e_i$

$= \lambda \alpha(F) + \alpha(G)$. ✓

injective: Suppose $\alpha(F) = 0$

Then $\sum_{i=1}^{3} f_i dx_i = 0$

$= \lambda\alpha(F) + \alpha(h)$. ✓

Suppose injective $\alpha(F) = 0$

Then $\sum_{i=1}^{3} f_i \, dx_i = 0$

So $\sum_{i=1}^{3} f_i(x) \, dx_i(x) = \underbrace{0}$ for $x \in \mathbb{R}^3$

$\hookrightarrow$ 0 alternating tensor acting on the tangent space of $\mathbb{R}^3$ at x; i.e. $0 \in T_x(\mathbb{R}^3)$

Then since $dx_1(x)$, $dx_2(x)$, & $dx_3(x)$ are the elementary dual elements, they are linearly independent hence $f_1(x) = f_2(x) = f_3(x) = 0$.

This holds for all x & so $f_i = 0 \ \forall i = 1, 2, 3$.

Hence $F = 0$ & so injective.

(x)

$T_x(\mathbb{R}^3))$

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Surjective

Let $w \in \Omega^1(\mathbb{R}^3)$. Then $w = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$

for some scalar functions. Then $\alpha(F) = w$

when F is a vector field s.t $F(x) = (x; \sum_{i=1}^{3} f_i(x) e_i)$

let $w \in \Omega^1(\mathbb{R}^3)$. Then $w = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$

for some scalar functions. Then $\alpha(F) = w$

when F is the vector field s.t $F(x) = (x; \sum_{i=1}^{3} f_i(x) e_i)$

ii) Claim: β is an isomorphism

linear. $\beta(\lambda F + G) = \sum_{i=1}^{3} (\lambda f_i + g_i) dx_{\text{no } i} (-1)^{i-1}$

$= \lambda \sum_{i=1}^{3} f_i \, dx_{\text{no } i} (-1)^{i-1} + \sum_{i=1}^{3} g_i \, dx_{\text{no } i} (-1)^{i-1}$

$= \lambda \beta(F) + \beta(G)$ ✓

Suppose

injective $\beta(F) = 0$

$\wedge \sum_{i=1} f_i \, dx_{no\,i} (-1)^{i-1} \to \sum_{i=1} g_i \, dx_{no\,i}$

$\wedge\beta(F) \to \beta(G)$

Suppose

injective $\wedge\beta(F) = 0$

$$\sum_{i=1}^{3} f_i \, dx_{no\,i} (-1)^{i-1} = 0$$

$$\sum_{i=1}^{3} f_i(x) \, dx_{no\,i}(x) (-1)^{i-1} = 0 \text{ for } x \in \mathbb{R}^3$$

↳ Once again since $dx_{no\,1}(x)$, $dx_{no\,2}(x)$, $dx_{no\,3}(x)$

are linearly independent alternating 2-tensors in $\Lambda^2(T_x(\mathbb{R}^3))$

then it follows $f_i(x) = 0 \ \forall i = 1, 2, 3$ & $\forall x$.

Hence $F = 0$ & so injective.

Surjective:

Let $w \in \Omega^2(\mathbb{R}^3)$

Then $w = h_1\, dx_{m1} + h_2\, dx_{m2} + h_3\, dx_{m3}$

for some scalar functions h_1, h_2, h_3.

Consider H vector field with

$$H(x) = (x;\ h_1(x)e_1 - h_2(x)e_2 + h_3(x)e_3)$$

Then
$$\beta(H) = \sum_{i=1}^{3} h_i\ (-1)^{i-1}(-1)^{i-1}\ dx_{mi}$$
$$= \sum_{i=1}^{3} h_i\ dx_{mi} = w.$$

$\square$

Step 2:

Show $\beta \circ \text{curl} = d \circ \alpha$.

Let $F(x) = (x; \sum_{i=1}^{3} f_i(x) e_i) \in V$.

Now $(\text{curl } F)(x) = (x; (D_2 f_3 - D_3 f_2) e_1 + (D_3 f_1 - D_1 f_3) e_2$
$+ (D_1 f_2 - D_2 f_1) e_3)$

Then $\beta(\text{curl}(F)) = (D_2 f_3 - D_3 f_2)\, dx_{\text{no } 1}$
$- (D_3 f_1 - D_1 f_3)\, dx_{\text{no } 2}$
$+ (D_1 f_2 - D_2 f_1)\, dx_{\text{no } 3}$

[illegible] MAT257 [illegible]

Now $(d \circ \alpha)(F)$

$$= d\left(\sum_{i=1}^{3} f_i \, dx_i\right)$$

$$= df_1 \wedge dx_1 + df_2 \wedge dx_2 + df_3 \wedge dx_3$$

$$= (D_2 f_1 \, dx_2 + D_3 f_1 \, dx_3) \wedge dx_1$$

$$+ (D_1 f_2 \, dx_1 + D_3 f_2 \, dx_3) \wedge dx_2$$

$$+ (D_1 f_3 \, dx_1 + D_2 f_3 \, dx_2) \wedge dx_3$$

$$+ (D_1 f_2\, dx_1 + D_3 f_2\, dx_3) \wedge dx_2$$

$$+ (D_1 f_3\, dx_1 + D_2 f_3\, dx_2) \wedge dx_3$$

$$= - D_2 f_1\, dx_1 \wedge dx_2 - D_3 f_1\, dx_1 \wedge dx_3$$

$$+ D_1 f_2\, dx_1 \wedge dx_2 - D_3 f_2\, dx_2 \wedge dx_3$$

$$+ D_1 f_3\, dx_1 \wedge dx_3 + D_2 f_3\, dx_2 \wedge dx_3$$

$$= dx_1 \wedge dx_2 \left[D_1 f_2 - D_2 f_1 \right]$$

$$+ dx_1 \wedge dx_3 \left[D_1 f_3 - D_3 f_1 \right]$$

$$+ D_1 f_3 \, dx_1 \wedge dx_3 + D_2 f_3 \, dx_2 \wedge dx_3$$

$$= dx_1 \wedge dx_2 \, [D_1 f_2 - D_2 f_1]$$

$$+ dx_1 \wedge dx_3 \, [D_1 f_3 - D_3 f_1]$$

$$+ dx_2 \wedge dx_3 \, [D_2 f_3 - D_3 f_2].$$

Hence $\beta \circ \text{curl} = d \circ \alpha$.

$$\text{So} \quad \boxed{\text{curl} = \beta^{-1} \circ d \circ \alpha.}$$

$\square$

$\boxed{64}$

$c \in C_2(\mathbb{R}^2)$ such that $c(r, \theta) = \begin{pmatrix} r\cos 2\pi\theta \\ r\sin 2\pi\theta \end{pmatrix}$

Then $c_{(1,0)}(t) = c \circ I^2_{(1,0)}(t)$

$= c(0, t)$

$= (0, 0)$

$$= (0, 0)$$

$$c_{(1,1)}(t) = c \circ I^2_{(1,1)}(t)$$

$$= c(1, t)$$

$$= (\cos 2\pi t, \sin 2\pi t)$$

$$c_{(2,0)}(t) = c \circ I^2_{(2,0)}(t)$$

$$= c(t, 0)$$

$$= (t \cos 0, t \sin 0)$$

$$= (t, 0)$$

$= (t\cos 0, t\sin 0)$

$= (t, 0)$

$$c_{(2,1)}(t) = c \circ I^2_{(2,1)}(t)$$

$$= c(t, 1)$$

$$= (t\cos 2\pi, t\sin 2\pi)$$

$$= (t, 0)$$

Then $\partial c = \sum_{i=1}^{2} \sum_{\alpha \in \{0,1\}} (-1)^{i+\alpha} c_{(i,\alpha)}$

$$\partial c(t) = -c_{(1,0)}(t) + c_{(1,1)}(t) + c_{(2,0)}(t) - c_{(2,1)}(t)$$

$$= -c_0(t) + c_1(t) + (t, c) - (t, c)$$

$$= c_1(t) - c_0(t).$$

Hence $dc = c_1 - c_0$. $\blacksquare$

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$\boxed{GS}$ $c: I^k \to \mathbb{R}^n$ & $w \in \Omega^k(\mathbb{R}^n)$ & $r: I^k \to I^k$

$$\int_{c \circ r} w = \int_{I^k} (c \circ r)^* w = \int_{I^k} r^* c^* w$$

Now $c^* w \in \Omega^k(I^k)$

& so $c^* w = f \, dx_1 \wedge \cdots \wedge dx_k$

$$= \int_{I^k} r^* (f \, dx_1 \wedge \cdots \wedge dx_k)$$

$$= \int_{I^k} r^* f \wedge r^* (dx_1 \wedge \cdots \wedge dx_k)$$

$$\int_{I^k} r^*(f\,dx_1\wedge\cdots\wedge dx_k)$$

$$= \int_{I^k} r^*f \wedge r^*(dx_1\wedge\cdots\wedge dx_k)$$

$$= \int_{I^k} f\circ r\,(\det Dr)\,dx_1\wedge\cdots\wedge dx_k$$

$$= \int_{[0,1]^k} f\circ r\,|\det Dr|$$

change of variables

$$= \int_{r([0,1]^k)} f$$

since r bijective

$$= \int_{[0,1]^k} f$$

$$= \int_{I^k} f\,dx_1\wedge\cdots\wedge dx_k$$

$$= \int_{I^k} c^*\omega$$

$$= \int_c \omega$$

$\square$