Q.1 Define a product of function $\phi_i * \psi_j : V \times W \to \mathbb{R}$

where $\phi_i * \psi_j (a_1 a_2 \cdots a_n , b_1 b_2 \cdots b_m)$ $a_i , b_i \in \mathbb{R}$

$= \phi_i (a_1 a_2 \cdots a_n) \cdot \psi_j (b_1, b_2 \cdots b_m)$

Similary to ⊗, but based on different domain

Moreover, by set-up and Given, $\phi_i * \psi_j (v_i, w_j) = 1$

if $m \neq i$ or $n \neq j$ $\phi_i * \psi_j (v_m, w_n) = 0$

NTS $\{\phi_i * \psi_j\}_{i \in \{1,2 \cdots n\},\ j \in \{1,2 \cdots m\}}$ be basis of B

by definition of $\phi_i * \psi_j$ it's bilinear

Firstly Let $b \in B$ be arbitrary

Let $b(v_i, w_j) = b_{ij} \in \mathbb{R}$ $\forall i,j$

Then $b = \sum_{i,j} b_{ij} \cdot (\phi_i * \psi_j)$

since $b_{ij} \in \mathbb{R} \Rightarrow \{\phi_i * \psi_j\}$ spans B

Moreover suppose $\sum_{i,j} a_{ij} (\phi_i * \psi_j) = 0$

Then $\left(\sum_{i,j} a_{ij} (\phi_i * \psi_j)\right)(v_i, w_j)$

$= \quad a_{ij} = 0$

Since i, j are arbitrary $\Rightarrow \{\phi_i * \psi_j\}$ are linearly indep.

$\Rightarrow \{\phi_i * \psi_j\}_{i,j}$ be basis of B

$\dim(B) = |\{\phi_j * \psi_j\}_{i,j}| = m \times n$

Q_2 $w = xy\,dx + 2x\,dy - y\,dz$

$a(u,v) = (uv, u^2, 3u+v)$

$a^* w = u^3 v\, d uv + 2uv\, du^2 - u^2\, d(3u+v)$

$= u^3 v\,(v\,du + u\,dv) + 2uv \cdot (2u\,du) - u^2 \cdot (3du + dv)$

$= u^3 v^2\,du + u^4 v\,dv + 4u^2 v\,du - 3u^2\,du - u^2\,dv$

$= (u^3 v^2 + 4u^2 v - 3u^2)\,du + (u^4 v - u^2)\,dv$

$dw = dx \wedge \frac{\partial w}{\partial x} + dy \wedge \frac{\partial w}{\partial y} + dz \wedge \frac{\partial w}{\partial z}$

since $dx \wedge dx \wedge dy = 0$ we just remove items like this.

$= dx \wedge (2dy) + dy \wedge (x\,dx) - dy \wedge dz + dz \wedge (0)$

$= (2-x)\, dx \wedge dy - dy \wedge dz$

$d(a^* w) = du \wedge \frac{\partial (a^* w)}{\partial u} + dv \wedge \frac{\partial (a^* w)}{\partial v}$

by above, remove items like $du \wedge du$, $dv \wedge dv$

$= du \wedge (4u^3 v - 2u)\,dv + dv \wedge (2u^3 v + 4u^2)\,du$

$= (4u^3 v - 2u - 2u^3 v - 4u^2)\, du \wedge dv$

$= (2u^3 v - 4u^2 - 2u)\, du \wedge dv$

$$a^*(dw) = (2-uv)\, duv \wedge du^2 - du^2 \wedge d(3u+v)$$

$$= (2-uv)(u\,dv + v\,du) \wedge (2u\,du) - (2u\,du) \wedge (3du + dv)$$

$$= 2u^2(uv-2)\, du \wedge dv - (2u)\, du \wedge dv$$

$$= (2u^3v - 4u^2 - 2u)\, du \wedge dv$$

$$= d(a^*w)$$

Q3. Curl maps vector fields $\xrightarrow{\text{curl}}$ vector fields

$$(F_1, F_2, F_3) \longmapsto \left(\underbrace{\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}}_{g_1},\ \underbrace{\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}}_{g_2},\ \underbrace{\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}}_{g_3} \right)$$

There is a bijection map, $P : (F_1, F_2, F_3) \to \Omega^1(\mathbb{R}^3)$

where $P(F_1, F_2, F_3) = F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3$

is 1-1 and onto

Moreover, there is a bijection map $H : (g_1, g_2, g_3) \to \Omega^2(\mathbb{R}^3)$

where $H(g_1, g_2, g_3) = g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2$

$\dim(\text{Range}(g_1, g_2, g_3)) = 3$ and also 1-1

Thus.

$$\begin{tikzcd}
\text{vector fields} \arrow[r, "\text{curl}"] \arrow[d, shift left, "\text{bijective } P"] & \text{vector fields} \arrow[d, shift left, "\text{bijective } H"] \\
\Omega^1(\mathbb{R}^3) \arrow[r, "d"] \arrow[u, shift left] & \Omega^2(\mathbb{R}^3) \arrow[u, shift left, "H^{-1}"]
\end{tikzcd}$$

$$dP = \frac{\partial F_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial F_1}{\partial x_3} dx_3 \wedge dx_1 + \frac{\partial F_2}{\partial x_1} dx_1 \wedge dx_2$$

$$+ \frac{\partial F_2}{\partial x_3} dx_3 \wedge dx_2 + \frac{\partial F_3}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial F_3}{\partial x_2} dx_2 \wedge dx_3$$

$$= g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2$$

$$= H$$

Therefore. curl is equivalent to $H^{-1} \circ d \circ P$

where $d : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$

In this way, curl arises as an instance of d

for $n=3$, $k=1$

$\square$

Q4.

Let $C(r,t) = (r\cos 2\pi t,\ r\sin 2\pi t)$ $\quad C \in C_2(\mathbb{R}^2)$

$$\partial C = \sum_{i=1}^{2} \sum_{\alpha\in\{0,1\}} (-1)^{i+\alpha} C_{(i,\alpha)}$$

$$= \sum_{i=1}^{2} \sum_{\alpha\in\{0,1\}} (-1)^{i+\alpha} C_{(i,\alpha)}$$

$$= -C_{[1,0]} + C_{[1,1]} + C_{[2,0]} - C_{[2,1]}$$

$$= -(0,0) + (\cos 2\pi t, \sin 2\pi t) + (r\cos 0, r\sin 0) - (r\cos 2\pi, r\sin 2\pi)$$

$$= (\cos 2\pi t, \sin 2\pi t) - (0,0) + (r,0) - (r,0)$$

$$= (\cos 2\pi t, \sin 2\pi t) - (0,0)$$

$= C_1 - C_0$ as we want $\square$

Q5. By Given and property in lecture,

$$LHS = \int_C w = \int_{I^k} c^* w$$

$$RHS = \int_{c \circ \gamma} w = \int_{I^k} (c \circ \gamma)^* w$$

$$= \int_{I^k} \gamma^* \circ c^* w$$

by Thm 4-9
$$= \int_{I^k} (c^* w \circ \gamma) \det(\gamma') \quad \cdots \text{①}$$

since $\forall p \in I^k$, $\det(\gamma'(p)) > 0$, γ is bijection between I^k and I^k

$\therefore$ γ maps $\det(\gamma'(p))$ to unique point in I^k

$\because$ Image $(\gamma) = I^k$ we know

by COV: $\text{①} = \int_{I^k} c^* w$

$$= \int_C w$$

$\Rightarrow$ RHS = LHS as we want $\square$