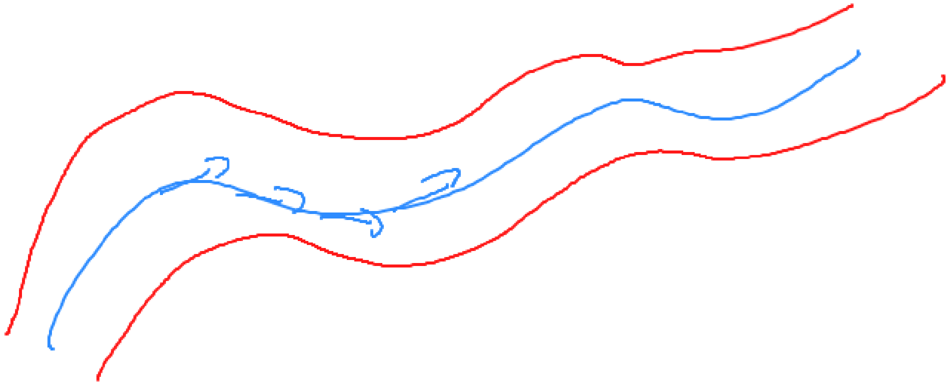
M is k-dim mfld

$F \in \mathrm{Vect}(M)$



$M \subset \mathbb{R}^n$ $\forall x \in M$ $U_x \subset \mathbb{R}^n$ "nice"

$A = \bigcup_{x \in M} U_x$

solve locally (for each U_x indiv.), then glue using p.o.u. subord. to $\{U_x\}_{x \in M}$.

local $\Leftrightarrow$ reduce this problem to: in local coord.

$$f_1 \frac{\partial}{\partial x_1} + \dots + f_k \frac{\partial}{\partial x_k} \qquad f_i : U_x \to \mathbb{R}$$

local fn-ns

$(x_1 \dots x_n)$ local coord.

Smooth $f: \mathbb{R}^m \to \mathbb{R}^n$.

$Df : \mathbb{R}^m \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^{m \times n}$

$D^2 f$

$\vdots$

19. Suppose $\omega = \partial \eta$

$$\int_{c_1} \partial \eta = \int_{c_2} \partial \eta \overset{\text{Stokes}}{\Rightarrow} \int_{\delta c_1} \eta = \int_{\delta c_2} \eta$$

$\delta c_1 = \delta c_2$

Now suppose ~~that~~ ω is precise

$c_i : [0,1] \to \mathbb{R}^n$

$$\int_{c_1 - c_2} \omega = \int_S \underset{\sim}{d\omega} = 0$$

$C : [0,1]^3 \to \mathbb{R}^7$

non-negative on $[0,1]^3$

$$0 = \int_{\underset{\sim}{C}} \omega = \int_{[0,1]^3} c^*\omega = \int_{[0,1]^3} \underset{\sim}{f}\, dx_1 dx_2 dx_3 = 0$$

3-vector $\quad f : \mathbb{R}^4 \to \mathbb{R}^3$

$xyzt \qquad xyz$

$$\Omega^0 \xrightarrow{d} \Omega^1$$

$$f \longrightarrow \underset{\shortparallel}{df}$$

$$\underbrace{\frac{\partial f}{\partial x} dx + \dots + \frac{\partial f}{\partial z} dz}+ \frac{\partial f}{\partial t} dt.$$

Scalar fns $\quad \mathbb{R}^4 \to \mathbb{R}$

3-vectors $\quad \mathbb{R}^4 \to \mathbb{R}^3$

grad

curl

div

For each fixed t grad, curl, div are the same grad $\quad F : \mathbb{R}^3 \to \mathbb{R} \Rightarrow \operatorname{grad} f : \mathbb{R}^3 \to \mathbb{R}^3$

$$(x,y,z) \to \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

$$\Omega^0 \to \Omega^1$$

$$f \to (\operatorname{grad} f, \partial_t)$$

$$F : \mathbb{R}^4 \to \mathbb{R} \overset{\text{same!}}{\Longleftrightarrow} F_t : \mathbb{R}^3 \to \mathbb{R}$$

(x,y,z,t)

$$\operatorname{grad} f(x,y,z,t) := \operatorname{grad} f_t(x,y,z)$$

curl $\quad G : \mathbb{R}^3 \to \mathbb{R}^3 \Rightarrow \operatorname{curl} G : \mathbb{R}^3 \to \mathbb{R}^3$

$$\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}, \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right).$$

$$\operatorname{curl} G(x,y,z,t) := \operatorname{curl} G_t(x,y,z)$$

15. (commutator)

$$[D_F, D_G] \overset{?}{=} D_H.$$

$$D_F \circ D_G + D_G \circ D_F \overset{?}{=} D_{\tilde{H}}$$

$\Downarrow$

will not work!

$(n-1)$- dim mfld in $\mathbb{R}^n$

Demo:

$$\left[f\frac{\partial}{\partial x}, g\frac{\partial}{\partial y}\right] = f\frac{\partial}{\partial x}\left(g\frac{\partial}{\partial y}\right) - g\frac{\partial}{\partial y}\left(f\frac{\partial}{\partial x}\right) =$$

$$= f g_x \frac{\partial}{\partial y} + fg\cancel{\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)} - g f_y \frac{\partial}{\partial x} - gf\cancel{\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x}\right)}$$

$$fg = gf$$

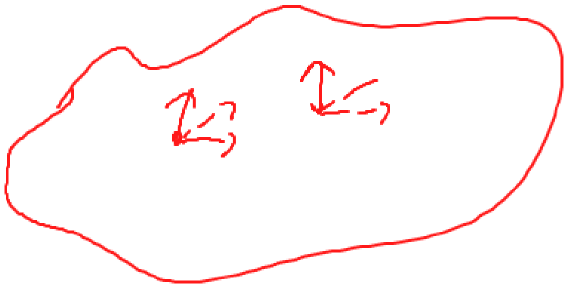
$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$$

commute!

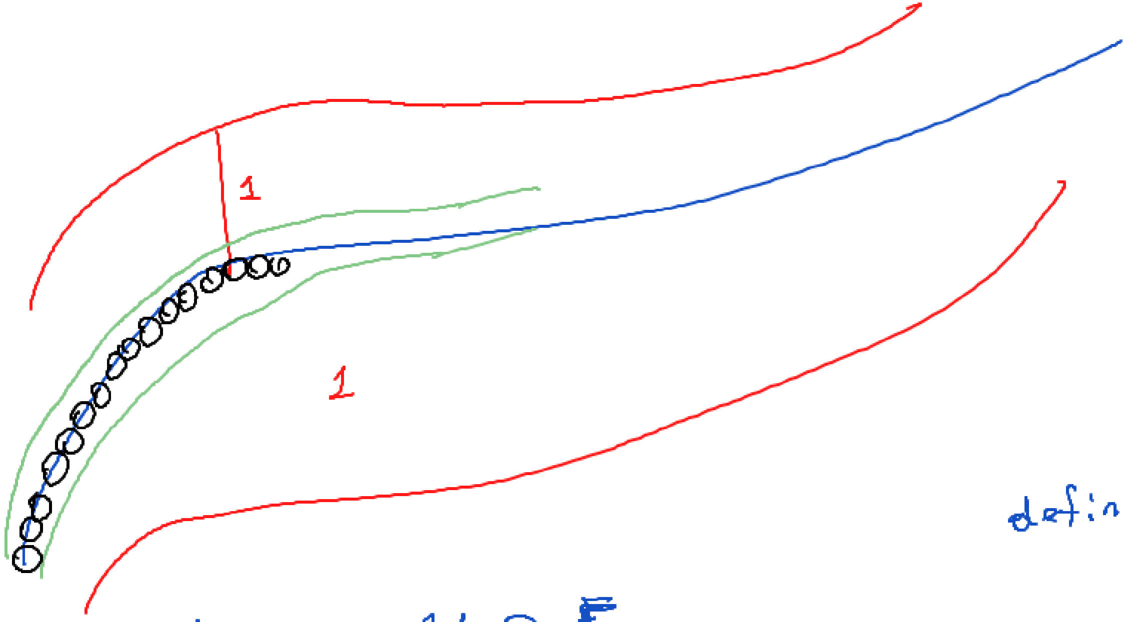
Thm: An $(n-1)$-dim mfld is orientable $\Leftrightarrow$ there is a non-van. normal v. f.

Proof: $(\Leftarrow)$ Let N be a normal field, ω is a volume form on $\mathbb{R}^n$. $\omega_M(v_1, \dots v_{n-1}) := \omega(v_1, \dots v_{n-1}, N)$

$(\Rightarrow)$ M is orientable.



As M is Riemannian, there is a consistent metric in all ~~tgt~~ spaces, so consider vectors orthogonal to local frames.



Urysohn's lemma

Let $\mathcal{U} \supset \Gamma_f$

ε-nbhd of graph

(don't consider 1_{Γ_f})

define $1_{\mathcal{U}}$.

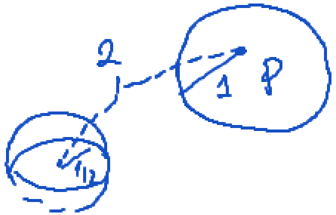
smooth on Γ_f itself.

g

$$h(x,y) = g(|y-f(x)|^2)$$

Thm: (div thm) $M \subset \mathbb{R}^3$ is compact 3-dim mfld w/ bndry, n is the unit normal on ∂M. F is diff v.f. on M.

$$\int_M \operatorname{div} F \, dV = \int_{\partial M} \langle F, n \rangle \, dA$$



$$\int_{\partial D^3_2} \langle E, n \rangle \, dA = \int_{D^3_2} \operatorname{div} E \, dV \overset{\substack{\operatorname{div} E = 0 \\ \text{outside of} \\ D^3_{1/2}}}{=} \int_{D^3_2 \cap D^3_{1/2} = D^3_{1/2}} \operatorname{div} E \, dv \overset{\text{div}}{=} \int_{\partial D^3_{1/2}} \langle E, n \rangle \, dA = 2\pi.$$

$$\int_{\partial D^3_1(p)} \langle E, n \rangle \, dA = \int_{D^3_1(p)} \operatorname{div} E \, dV = \underline{\int_{D^3_1(p) \cap D^3_{1/2}} \operatorname{div} E \, dV} = 0.$$

$p \in \partial D^3_2$

$\|\cdot\|$

X is Banach space!

A is Banach algebra if

$\|ab\| \leq \|a\| \cdot \|b\|$

M is a Banach module over banach algebra if M is a module, $A \times M \to M$ is continuous

Recall: Let R be a ring (algebra)

M is an R module if there is a "nice" operation

$* : R \times M \to M$

$\underset{\sim}{a}(\underset{\sim}{m} + \underset{\sim}{n}) = am + an$

$ab(m) = a(bm)$

$\vdots$