$f \circ T_\theta = f$

$f(T_\theta(a)) = f(a) \Rightarrow f(e^{i\theta} a) = f(a) \quad \forall \theta \in [0, 2\pi]$.

$D(f \circ T_\theta)(a) = Df(T_\theta(a)) \circ DT_\theta(a) = D(f)(a)$

$DT_\theta(a) = T_\theta$

differential of a linear transform equals the transf. itself!

$D(f \circ T_\theta)(a) \begin{pmatrix} -y \\ x \end{pmatrix} = \dots$

$f(x, y) = |xy|$

If $x_0 \neq 0$, $y_0 \neq 0$ $\Rightarrow$ f is diffable at (x_0, y_0) $\Rightarrow$

$\frac{\partial f}{\partial x} = \pm |y_0|$, $\frac{\partial f}{\partial y} = \pm |x_0|$

$(x_0 \neq 0)$

$\frac{\partial f}{\partial x}(x_0, y_0) = \mathrm{sgn}(x_0)|y_0| = \lim_{h \to 0} \frac{|x_0 + h||y_0| - |x_0 y_0|}{h}$

$\frac{\partial f}{\partial y}(x_0, y_0) = \mathrm{sgn}(y_0)|x_0|$

$(y_0 \neq 0)$

$f = |xy|$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{|x_0+h||y_0| - |x_0||y_0|}{h} = |y_0| \cdot \underbrace{\lim_{h\to 0} \frac{|x_0+h| - |x_0|}{h}}_{\begin{cases} 1 & \text{if } x_0 > 0 \\ -1 & \text{if } x_0 < 0 . \end{cases}} = \operatorname{sgn}(x_0)\, |y_0|$$

$-1 \longrightarrow \bullet \longleftarrow 1$

$(0,1)$

a discontinuity!

Let us try doing the same for $x_0 = 0$.

$$\lim_{h\to 0} \frac{|h||y_0| - |0||y_0|}{h} = \lim_{h\to 0} \frac{|h|\,|y_0|}{h} = \lim_{h\to 0} \operatorname{sgn}(h)\cdot|y_0|$$

this limit does not make sense for $x_0 = 0$, $y_0 \neq 0$.

However, we claim that $|xy| = o\left(\sqrt{x^2+y^2}\right)$ (we guess that $Df(0)=0$)

$$\lim_{(x,y)\to 0} \frac{|xy|}{\sqrt{x^2+y^2}} = 0 \iff \lim_{(x,y)\to 0} \frac{x^2y^2}{x^2+y^2} = 0 = \lim_{(x,y)\to 0} |xy| \boxed{\frac{|xy|}{x^2+y^2}}$$

$|xy| \leq x^2+y^2$

bounded times $\searrow 0 = \searrow 0$

TT1R5:

$$f = o(h)$$

$$\Downarrow$$

$$f(0) = 0.$$

$$\Downarrow$$

$$\underset{\underset{0}{\shortparallel}}{f(h) - f(0)} = \underset{\underset{0}{\shortparallel \text{ def}}}{Df(0)}(h) + o(h)$$

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

TT1R6:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} e^{x} + e^{y} \\ e^{x} + e^{-y} \end{pmatrix}$$

$$\det \begin{pmatrix} e^{x_0} & e^{y_0} \\ e^{x_0} & -e^{-y_0} \end{pmatrix} = \underbrace{-e^{x_0 - y_0}}_{<0} \underbrace{- e^{x_0 + y_0}}_{<0}$$

$$(e^{-x})' = -e^{-x}$$

4: $x_1 \rightsquigarrow \lambda x_1$

$x_2 \rightsquigarrow \lambda x_2$ $\longrightarrow$ WLOG $\lambda = Id$. . . .

Consider $\theta = \pi$.

$f(T_\pi(x)) = f(-x)$

$Df(-x)) = -Df(x)$

$f(x) = f(-x)$

$Df_x = -Df_{-x}$

$$Df_{T_\theta a} \circ T_\theta = Df_a$$

$$\left(\frac{\partial f}{\partial x}\Big|_{T_\theta a}, \frac{\partial f}{\partial y}\Big|_{T_\theta a} \right) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \left(\frac{\partial f}{\partial x}\Big|_a, \frac{\partial f}{\partial y}\Big|_a \right)$$

Argue in terms of directional derivatives!

Consider $g_a : \theta \mapsto f(T_\theta(a))$. g_a is a constant f-n $g: \mathbb{R} \to \mathbb{R}$

On the one hand, $g_a'(\theta) = 0 \quad \forall \theta \in \mathbb{R}$.

apply the chain rule to this

chain rule

$$\underset{\mathbb{R}}{\theta} \xrightarrow{T_\theta} \underset{\mathbb{R}^2}{T_\theta(a)} \xrightarrow{f} \underset{\mathbb{R}}{f(T_\theta(a))}$$

$$\left(\frac{\partial f}{\partial x}\Big|_{T_0(a)}, \frac{\partial f}{\partial y}\Big|_{T_0(a)} \right) \circ \begin{pmatrix} -\sin 0 & -\cos 0 \\ \cos 0 & -\sin 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$Df|_a \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0.$$

Recall:

a homogeneous fn-n : $f(\lambda x) = \lambda^n f(x)$

$$\frac{d}{d\lambda} f(\lambda x)$$

$x \rightsquigarrow x^3$ is 1-1

$(x^3)' = 3x^2$ (for $x = 0$ $\cancel{\neq}\, 0$)

If you have a fn-n $f: \mathbb{R}^n \to \mathbb{R}^n \Rightarrow$

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \Rightarrow \det Df \text{ is well-def.}$$

Jacobi matrix

Jacobian

arctan (x)

$f: \mathbb{R}^n \to \mathbb{R}^p$

$p \le n$

$f(a) = 0$

$$n\Bigg\{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_p}{\partial x_1} & & \frac{\partial f_p}{\partial x_n} \end{pmatrix}\Bigg\}p$$

rk is full $\Rightarrow$

$\Rightarrow$ p lin. indep. columns $\Rightarrow$

det of the minor is non-zero $\Rightarrow$

$\Rightarrow$ IFT.

11: try

$x \mapsto \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$

$x, x + 2\pi$

z is a function of x, y

$$1 + \frac{\partial z}{\partial x} = \cos(xyz) \cdot \frac{\partial}{\partial x}(xyz) =$$

$$= \cos(xyz)\left(yz + xy\frac{\partial z}{\partial x}\right)$$

$\exists\, g(x,y)$:

$x + y + g(x,y) = \sin(xy\,g(x,y))$

$\frac{\partial}{\partial x}$: $1 + \frac{\partial g}{\partial x} = \cos$

express $\frac{\partial z}{\partial x}$ in terms of x, y, z and then plug in $x = y = z = 0$