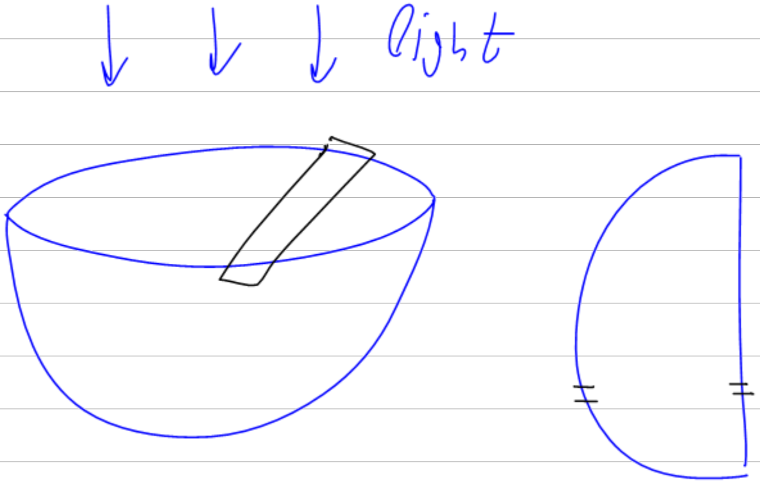
if $\pi = \frac{a}{b}$, consider n large

$$0 < \underbrace{\int_0^{\pi} \underbrace{\frac{x^n(a-bx)^n}{n!}}_{\mathbb{Z}} \underbrace{\sin x}_{\mathbb{Z}} \, dx}_{\mathbb{Z}} < 1$$

$\dot{\gamma} = \frac{d}{dt}\gamma$


light

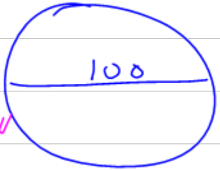
MAT257 Analysis II on April 12, 2021: the Maxwell equations. Last class of the year!
Read Along: These notes!
No Petr tutorial today!

Our Final Assessment will take place on Crowdmark (as the Term Tests) on Tuesday April 20 at 9AM. A detailed pre-exam office hours schedule will be sent later today or tomorrow. A bit later I will post some "reject questions".

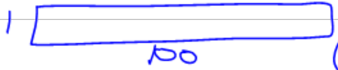
The material: *everything*, with a small bias in favour of the later stuff.

How to study? First, ***understand***. Only after, solve problems.

Can you cover



with 99



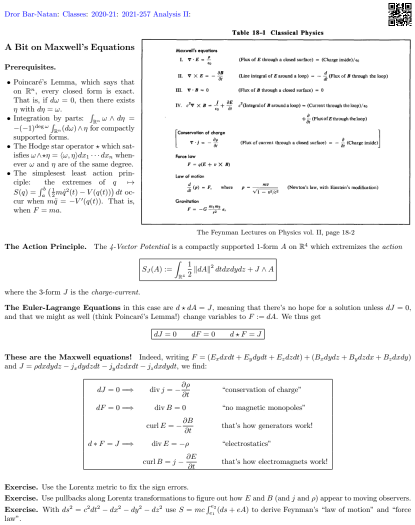
?

"Not enough shadow"

Yesterday I saw horses of more than one colour!

A Bit on Maxwell's Equations

There's also a handout at http://drorbn.net/2021-257/ap/Maxwell.pdf



A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg\omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle\omega, \eta\rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2}m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18–1 Classical Physics

Maxwell's equations

I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$ (Flux of E through a closed surface) = (Charge inside)/ϵ_0

II. $\nabla \times E = -\frac{\partial B}{\partial t}$ (Line integral of E around a loop) = $-\frac{d}{dt}$(Flux of B through the loop)

III. $\nabla \cdot B = 0$ (Flux of B through a closed surface) = 0

IV. $c^2\nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$ c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+\frac{\partial}{\partial t}$(Flux of E through the loop)

[Conservation of charge
$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$(Charge inside)]

Force law
$F = q(E + v \times B)$

Law of motion
$\frac{d}{dt}(p) = F$, where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation
$F = -G\frac{m_1 m_2}{r^2}e_r$

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2}\|dA\|^2\, dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\operatorname{div} j = -\frac{\partial\rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = j - \frac{\partial E}{\partial t}$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".

Table 18-1 Classical Physics

Maxwell's equations

I. $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$ — (Flux of $\boldsymbol{E}$ through a closed surface) $=$ (Charge inside)/ϵ_0

II. $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ — (Line integral of $\boldsymbol{E}$ around a loop) $= -\frac{d}{dt}$ (Flux of $\boldsymbol{B}$ through the loop)

III. $\nabla \cdot \boldsymbol{B} = 0$ — (Flux of $\boldsymbol{B}$ through a closed surface) $= 0$

IV. $c^2 \nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{\epsilon_0} + \frac{\partial \boldsymbol{E}}{\partial t}$ — c^2(Integral of $\boldsymbol{B}$ around a loop) $=$ (Current through the loop)/ϵ_0 $+ \frac{\partial}{\partial t}$ (Flux of $\boldsymbol{E}$ through the loop)

recover

Conservation of charge

$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t}$ — (Flux of current through a closed surface) $= -\frac{\partial}{\partial t}$ (Charge inside)

Force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

within easy reach

Law of motion

$\frac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}$, where $\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\boldsymbol{e}_r$$

The Feynman Lectures on Physics vol. II, page 18-2

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Prerequisite 1.

Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.

Prerequisite 2.

$$\int f' g = -\int f g' \pm \text{bndry terms}$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta \pm \omega \wedge d\eta$$

Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.

$$0 = \int_{\partial B} \omega \wedge \eta = \int_{\mathbb{R}^n} d(\omega \wedge \eta) = \int_{\mathbb{R}^n} d\omega \wedge \eta \pm \int_{\mathbb{R}^n} \omega \wedge d\eta$$

Prerequisite 3.

$$\binom{n}{k} = \binom{n}{n-k}$$

$dx \longleftrightarrow dy \wedge dz$
$dy \longleftrightarrow dz \wedge dx$

$$\cancel{dx_I \longleftrightarrow dx_{I^c}}$$

naive.

The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.

Start from O.N. v_i
consider the dual basis (φ_i)

$$\omega \wedge (*\omega) = |\omega|^2 dx_1 \ldots dx_n$$

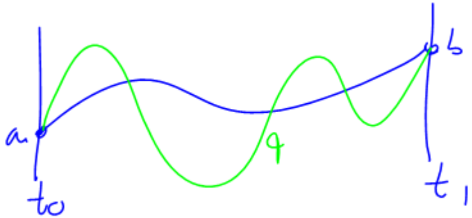
$$\langle \varphi_I, \varphi_J \rangle = \delta_{IJ}$$

this is indep of the ON basis

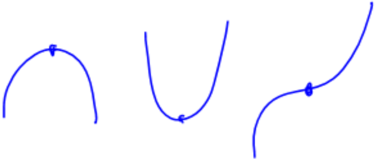
$\mathbb{R}^4_{t,x,y,z}$

$$*(dt \wedge dx) = \pm dy \wedge dz$$

Prerequisite 4.



The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_{a\ t_0}^{b\ t_1} \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $\underbrace{m\ddot{q} = -V'(q(t))}_{\text{Euler-Lagrange}}$. That is, when $F = ma$.



The Action Principle. For E&M

$$A \in \Omega^1_{\text{Compact}}(\mathbb{R}^4)$$

The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} |dA|^2 \, dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*. $J \in \Omega^3_c(\mathbb{R}^4)$

$$J = \rho \, dxdydz - j_x \, dydzdt - j_y \, dzdxdt - j_z \, dxdydt$$

charge. current.

$$\Updownarrow$$

$\forall B$

$$S_J(A+\epsilon B) = S_J(A) + \epsilon \underbrace{\cancel{\quad\quad}}_{\text{vanish}} + \dots$$

$$f(a+\epsilon b)$$
$$= f(a) + \cancel{\quad}\,\epsilon + o(\epsilon)$$

$$S_J(A+\epsilon B) = \int_{\mathbb{R}^4} \frac{1}{2}\langle d(A+\epsilon B), d(A+\epsilon B)\rangle + J\wedge(A+\epsilon B)$$

$$= \int_{\mathbb{R}^4} \frac{1}{2}\langle dA, dA\rangle + J\wedge A + \epsilon\left(\frac{2}{2}\int \langle dB, dA\rangle + J\wedge B\right) + \cancel{\epsilon^2}$$

A is extremal $\iff \forall B \quad \int \langle dB, dA\rangle + J\wedge B = 0$

$$\Leftrightarrow 0 = \int_{\mathbb{R}^4} dB \wedge (*dA) + J \wedge B = 0$$

$$= \int +B \wedge (d*dA) - B \wedge J = \int B \wedge (\underbrace{d*dA - J})$$

$$\Leftrightarrow d*dA - J = 0$$

$$\Leftrightarrow d*\underbrace{dA}_{F} = J$$

Can only have sol'ns if $dJ = 0$.

$dJ = 0 \quad d{*}F = J \quad dF = 0$

The Euler-Lagrange Equations

in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations!

"Magnetic Field"

Electric Field

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

Writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\operatorname{div} j = -\dfrac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
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$dJ = 0$

$J \in \Lambda^3 \quad dJ \in \Lambda^4$

$$dJ = \Big(\qquad\qquad \Big)\, dt\, dx\, dy\, dz$$

$= 0$

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d\star F = J}$$

with $J = \rho\, dxdydz - j_x\, dydzdt - j_y\, dzdxdt - j_z\, dxdydt$.

$$\boxed{dJ = 0 \implies \operatorname{div} j = -\frac{\partial \rho}{\partial t} \qquad \text{“conservation of charge”}}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0$$

$dF = 0$

$dF = (\quad)\, dx\,dy\,dz + (\quad)\, dt\,dy\,dz$
$(\quad)\, dt\,dz\,dx$
$(\quad)\, dt\,dx\,dy$

scalar eq'n

Vector eq'n

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.

$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\dfrac{\partial B}{\partial t}$	that's how generators work!

$d * F = J$

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.
With $\omega \wedge *\omega = |\omega|^2 dtdxdydz$ we have

$$\left.\begin{array}{lll} *dxdt = -dydz, & *dydt = -dzdx, & *dzdt = -dxdy, \\ *dydz = -dxdt, & *dzdx = -dydt, & *dxdy = -dxdt, \end{array}\right\} \text{change}$$

so $*F = (-B_x dxdt - B_y dydt - B_z dzdt) + (-E_x dydz - E_y dzdx - E_z dxdy)$.

$d * F = J \Longrightarrow$	$\text{div } E = -\rho$	"electrostatics"
	$\text{curl } B = j - \dfrac{\partial E}{\partial t}$	that's how electromagnets work!

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III.	$\nabla \cdot \boldsymbol{B} = 0$	(Flux of $\boldsymbol{B}$ through a closed surface) = 0
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[**Conservation of charge**

$$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t}$$

(Flux of current through a closed surface) = $-\dfrac{\partial}{\partial t}$ (Charge inside)]

Force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Law of motion

$$\frac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}, \quad \text{where} \quad \boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$$

(Newton's law, with Einstein's modification)

Gravitation

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\boldsymbol{e}_r$$

Feynman again. But wait, in our last two equations the sign of E is wrong!

Exercise 1.

Use the Lorentz metric to fix the sign errors.

Metric should be $x^2+y^2+z^2-t^2$

not $x^2+y^2+z^2+t^2$

Exercise 2.

Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise 3.

With $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".