Dror Bar-Natan: Classes: 2020-21: 2021-257 Analysis II:



A Bit on Maxwell's Equations

There's also a handout http://drorbn.net/ 2021-257/ap/Maxwell.pdf

Dror Bay, Nature Classes: 2020, 21: 2021, 257 Analysis III



A Bit on Maxwell's Equations

Prerequisites.

- · Poincaré's Lemma, which says that on Ro, every closed form is exact. That is if $d\omega = 0$ then there exists η with $d\eta = \omega$.
- Integration by parts: $\int_{\Omega_n} \omega \wedge d\eta =$ $-(-1)^{\text{deg}\,\omega} \int_{\Omega_n} (d\omega) \wedge \eta$ for compactly supported forms.
- . The Hodge star operator * which satisfies $\omega \wedge *n = (\omega, n)dx_1 \cdots dx_n$ when-
- ever -: and n are of the same dorrer • The simplesest least action principle: the extremes of $q \mapsto$ $S(q) = \int_{0}^{q} (\frac{1}{2}m\dot{q}^{2}(t) - V(q(t))) dt$ occur when $m\tilde{a} = -V'(a(t))$. That is,
 - when F = ma.

Table 18-1 Classical Physics

Manager and America L V-E - A (Plus of E shough a closed surface) = (Charge inside)/s₀ B X X X - - # (Line integral of E around a loop) $= -\frac{d}{2}$ (Flux of E through the loop) III. V : # ~ 0 (Flor of # through a should surface) = 0

19. At 10 K - A + H Addressed Respectations - Cornel Security Section (1) $+\frac{\partial}{\partial t}$ (Plan of Ethrough the loop)

(Plax of current through a closed surface) = $-\frac{\partial}{\partial t}$ (Charge inside) F - 45 + + × 5 $\frac{d}{dt}(p) = F$, where $p = \frac{mp}{\sqrt{1-m^2/2}}$ (Newton's law, with Electric's modification) F = -0 DD +

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The 4-Vector Potential is a compactly supported 1-form A on \mathbb{R}^4 which extremizes the action

$$S_J(A) := \int_{\mathbb{R}^d} \frac{1}{2} ||dA||^2 dt dx dy dz + J \wedge A$$

where the 3-form, I is the charge-current

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless dJ = 0. and that we might us well (think Poincaré's Lemma!) change variables to F := dA. We thus set

$$dJ = 0$$
 $dF = 0$ $d \star F = J$

These are the Maxwell conations! Indeed, writing F = (E, dvdt + E, dudt + E, dvdt) + (B, dvdv + B, dvdv + B, dvdv)and $J = advdudz = i \cdot dudzdt = i \cdot dzdzdt = i \cdot dzdudt$, we find

$dJ = 0 \Longrightarrow$	$\operatorname{div} j = -\frac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d*F=J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = j - \frac{\partial E}{\partial t}$	that's how electromagnets work!

Propeles. Use the Lecentry metric to fly the sign course

Exercise. Use millbacks along Lorentz transformations to figure out how E and B (and i and a) amount to maring observers Exercise. With $ds^2 = c^2dt^2 - dx^2 - dx^2 - dx^2 - dx^2$ use $S = mc \int_{-\infty}^{\infty} (ds + eA)$ to derive Feynman's "law of motion" and "force

Table 18-1 Classical Physics

Maxwell's equations I. $\nabla \cdot E = \frac{\rho}{\epsilon}$ (Flux of E through a closed surface) = (Charge inside)/ ϵ_0 II. $\nabla \times E = -\frac{\partial B}{\partial t}$ (Line integral of E around a loop) $= -\frac{d}{dt}$ (Flux of B through the loop) (Flux of B through a closed surface) = 0 IV. $c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$ c^2 (Integral of B around a loop) = (Current through the loop)/ ϵ_0 $+\frac{\partial}{\partial x}$ (Flux of E through the loop) Conservation of charge $\nabla \cdot j = -\frac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) $= -\frac{\partial}{\partial t}$ (Charge inside) Force law $F = q(E + v \times B)$ Law of motion $\frac{d}{dt}(p) = F$, where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification) Gravitation $F = -G \frac{m_1 m_2}{r^2} e_r$

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Prerequisites.

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- ▶ Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.
- ▶ The Hodge star operator \star which satisfies $\omega \wedge \star \eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q\mapsto S(q)=\int_a^b\left(\frac{1}{2}m\dot{q}^2(t)-V(q(t))\right)dt$ occur when $m\ddot{q}=-V'(q(t))$. That is, when F=ma.

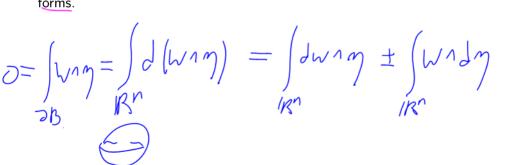
Prerequisite 1.

Poincaré's Lemma, which says that on \mathbb{R}^n , every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.

Prerequisite 2.

$$\frac{2}{d(wnn)} = \frac{1}{dwhn} \pm \frac{1}{dwnn}$$

Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.



3.
$$\binom{n}{k} = \binom{n}{n-k}$$

$$dx_{1} = dx_{1}$$

dx endende

The Hodge star operator \star which satisfies $\omega \wedge \star \eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.

 $W_1(*W) = |W|^2 dx_1 dx_1$

this is inex or the on busis

Prerequisite 4.

The simplesest least action principle: the extremes of $q\mapsto S(q)=\int_a^b\left(\frac{1}{2}m\dot{q}^2(t)-V(q(t))\right)dt$ occur when $m\ddot{q}=-V'(q(t))$. That is, when F=ma.

The Action Principle.

The 4-Vector Potential is a compactly supported 1-form A on \mathbb{R}^4 which extremizes the action

$$S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} |dA|^2 dt dx dy dz + J \wedge A$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations

in this case are $d \star dA = J$, meaning that there's no hope for a solution unless dJ = 0, and that we might as well (think Poincaré's Lemma!) change variables to F := dA. We thus get

$$dJ = 0$$
 $dF = 0$ $d \star F = J$

These are the Maxwell equations!

$$dJ = 0$$
 $dF = 0$ $d \star F = J$

Writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$$dJ=0 \Longrightarrow \quad \text{div}\,j=-\frac{\partial\rho}{\partial t} \qquad \text{"conservation of charge"}$$

$$dF=0 \Longrightarrow \quad \text{div}\,B=0 \qquad \text{"no magnetic monopoles"}$$

$$\text{curl}\,E=-\frac{\partial B}{\partial t} \qquad \text{that's how generators work!}$$

$$d*F=J \Longrightarrow \quad \text{div}\,E=-\rho \qquad \text{"electrostatics"}$$

$$\text{curl}\,B=j-\frac{\partial E}{\partial t} \qquad \text{that's how electromagnets work!}$$

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$$d * F = J$$

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 $dF = 0$ $d \star F = J$

with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.

With $\omega \wedge *\omega = |\omega|^2 dt dx dy dz$ we have

$$*dxdt = -dydz, \qquad *dydt = -dzdx, \qquad *dzdt = -dxdy,$$
 $*dydz = -dxdt, \qquad *dzdx = -dydt, \qquad *dxdy = -dxdt,$ so $*F = (-B_x dxdt - B_y dydt - B_z dzdt) + (-E_x dydz - E_y dzdx - E_z dxdy).$

$$d*F=J\Longrightarrow \qquad {
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Table 18-1 Classical Physics

Maxwell's equations

I.
$$\nabla \cdot E = \frac{\rho}{\epsilon}$$
 (Flux of E through a closed surface) = (Charge inside)/ ϵ_0

II.
$$\nabla \times E = -\frac{\partial B}{\partial t}$$
 (Line integral of E around a loop) = $-\frac{d}{dt}$ (Flux of B through the loop)

III.
$$\nabla \cdot \mathbf{B} = 0$$
 (Flux of \mathbf{B} through a closed surface) = 0

IV.
$$c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t} - c^2 (\text{Integral of } B \text{ around a loop}) = (\text{Current through the loop})/\epsilon_0 + \frac{\partial}{\partial t} (\text{Flux of } E \text{ through the loop})$$

$$\nabla \cdot f = -\frac{\partial \rho}{\partial t}$$
 (Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$ (Charge inside)

Force law

$$F = q(E + v \times B)$$

Law of motion

$$\frac{d}{dt}(p) = F$$
, where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation

vitation
$$F = -G \frac{m_1 m_2}{r^2} e_r$$

Feynman again. But wait, in our last two equations the sign of E is wrong!

Exercise 1.

Use the Lorentz metric to fix the sign errors.

Exercise 2.

Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise 3.

With $ds^2=c^2dt^2-dx^2-dy^2-dz^2$ use $S=mc\int_{e_1}^{e_2}(ds+eA)$ to derive Feynman's "law of motion" and "force law".