This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 11

Due: Wednesday January 27, 2021 11:59 PM (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. They are all taken from Munkres' *Analysis on Manifolds*, page 151. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Let V be a vector space of dimension n. Define a map $\iota\colon V \to (V^*)^*$ from V to the dual space of the dual space of V as follows. For $v \in V$ and $\phi \in V^*$, set

$$\iota(v)(\phi) = \phi(v).$$

(a) If $(v_1, \ldots, v_n) \in V^n$ is a basis of V and $(\phi_1, \ldots, \phi_n) \in (V^*)^n$ is its dual basis, show that $(\iota(v_1), \ldots, \iota(v_n))$ is a basis of $(V^*)^*$, which is in fact dual to the basis $(\phi_1, \ldots, \phi_n)$ of V^*.

(b) Deduce that ι is an isomorphism of vector spaces.

Q2 (10 points)

Let V be the vector spaces of polynomials p of degree ≤ 2 with coefficients in $\mathbb{R}$. Let $\phi_{-1}, \phi_0, \phi_1$ be the elements of V^* defined as follows:

$$\phi_x(p) = p(x) \qquad \text{for } x = -1, 0, 1.$$

Show that $\gamma = (\phi_{-1}, \phi_0, \phi_1)$ is a basis of V^* and find a basis β of V whose dual is γ (namely, such that $\beta^* = \gamma$).

Q3 (10 points)

Let V be a finite dimensional vector space and let $(v_1, \ldots, v_n)$ be a basis of V. Let k be a natural number. Define a map $B: \mathcal{T}^k(V) \times \mathcal{T}^k(V) \to \mathbb{R}$ as follows:

$$B(T_1, T_2) := \sum_{i_1, \ldots, i_k = 1}^{n} T_1(v_{i_1}, \ldots, v_{i_k}) T_2(v_{i_1}, \ldots, v_{i_k}).$$

(a) Show that $B \in \mathcal{T}^2(\mathcal{T}^k(V))$ (what does this even mean??).

(b) Show that B is an inner product. Namely, show that it is symmetric $B(T_1, T_2) = B(T_2, T_1)$, that for any T we have $B(T, T) \geq 0$, and that $B(T, T) = 0$ iff $T = 0$.