Homework Assignment 8

Due: Wednesday November 25, 2020 11:59 PM (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Spivak's book, pages 56 and 61-62. Note that Question 5 is worth double of the other questions! Note also that the late policy is strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Recall from reading the text that a set C is called "Jordan-measurable" if it is bounded and its boundary has measure 0, or equivalently, if it is bounded and its characteristic function is integrable in some rectangle that contains C.

Spivak's 3-22. If A is a Jordan-measurable set and $\epsilon > 0$, show that there is a compact Jordan-measurable set $C \subset A$ such that the volume of $A \setminus C$ is less than ϵ.

Q2 (10 points)

Recall from page 26 of the text that if f is a real-valued function, then $D_i f$ denotes its ith partial derivative, and $D_{i,j} f$ denotes the jth partial derivative of its ith partial derivative:

$$D_{i,j} f = D_j(D_i f)$$

(assuming all these quantities exist).

Spivak's 3-28. Use Fubini's theorem to give an easy proof that $D_{1,2} f = D_{2,1} f$, if these are continuous (hint in text).

Q3 (10 points)

Spivak's 3-32. Let $f\colon [a,b] \times [c,d] \to \mathbb{R}$ be continuous and suppose $D_2 f$ is continuous. Define $F(y) = \int_a^b f(x,y)dx$. Prove *Leibnitz' rule:* $F'(y) = \int_a^b D_2 f(x,y)dx$. (Hint in text).

Q4 (10 points)

Spivak's 3-34 (modified). Let $g_1, g_2\colon \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and suppose $D_1 g_2 = D_2 g_1$. Let

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

Show that $D_1 f = g_1$ and $D_2 f = g_2$. (Hint: Use the previous question).

Q5 (20 points)

Spivak's 3-35 (modified). Let $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear transformation of one of the following types:

$$\text{For some } 1 \le j \le n \text{ and some } \alpha \in \mathbb{R}, \qquad L(e_k) = \begin{cases} \alpha e_k & \text{if } j = k \\ e_k & \text{otherwise} \end{cases}$$

$$\text{For some } 1 \le i, j \le n, \qquad L(e_k) = \begin{cases} e_j & \text{if } i = k \\ e_i & \text{if } j = k \\ e_k & \text{otherwise} \end{cases}$$

$$\text{For some } 1 \le i, j \le n, \qquad L(e_k) = \begin{cases} e_i + e_j & \text{if } i = k \\ e_k & \text{otherwise} \end{cases}$$

(In all cases L is defined by its values on the standard basis vectors e_i).

(a) Show that if a set A is Jordan-measurable then so is $L(A)$.

(b) If v denotes the volume functional, show that $v(L(A)) = |\det L| \cdot v(A)$.

(c) Show that the formula from (b) holds for *any* linear transformation L, not necessarily of one of the three types above (hint in text).