Homework Assignment 7

Due: Wednesday November 18, 2020 11:59 PM (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Spivak's book, pages 49 and 52. Note that the late policy is strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 3-3. Let A be a rectangle in $\mathbb{R}^n$ and let $f, g\colon A \to \mathbb{R}$ be integrable.

(a) For any partition P of A and any subrectangle S, show that

$$m_S(f) + m_S(g) \leq m_S(f+g) \quad \text{and} \quad M_S(f+g) \leq M_S(f) + M_S(g)$$

and therefore

$$L(f,P) + L(g,P) \leq L(f+g,P) \quad \text{and} \quad U(f+g,P) \leq U(f,P) + U(g,P).$$

(b) Show that $f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.

(c) For any constant c, show that cf is integrable and $\int_A cf = c\int_A f$.

Q2 (10 points)

Spivak's 3-4. Let $f\colon A \to \mathbb{R}$ and let P be a partition of A. Show that f is integrable if and only if for each subrectangle S the function $F|_S$, the restriction of f to S, is integrable, and that in this case, $\int_A f = \sum_{S \in P} \int_S f|_S$.

Q3 (10 points)

Spivak's 3-5. Let $f, g\colon A \to \mathbb{R}$ be integrable and suppose $f \leq g$. Show that $\int_A f \leq \int_A g$.

Q4 (10 points)

Spivak's 3-6. If $f\colon A \to \mathbb{R}$ is integrable, show that $|f|$ is integrable and $\left|\int_A f\right| \leq \int_A |f|$.

Q5 (10 points)

Spivak's 3-9. (a) Show that an unbounded set cannot have content 0.

(b) Give an example of a closed set of measure 0 which does not have content 0.

Q6 (10 points)

Spivak's 3-11. Let $A = \bigcup_{i=1}^{\infty} (a_i, b_i)$ be a countable union of open intervals, and assume that $([0,1] \cap \mathbb{Q}) \subset A$. Show that if $\sum_{i=1}^{\infty} (b_i - a_i) < 1$ then the boundary of A is not of measure 0.

Q7 (10 points)

Spivak's 3-12. Let $f \colon [a, b] \to \mathbb{R}$ be an increasing function. Show that the set of discontinuities of f is of measure 0. (Hint in text).