Homework Assignment 6

Due: Wednesday October 28, 2020 11:59 PM (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Munkres' book, pages 78 and 79. Note that the late policy remains strict - you will lose 10% for each hour that you are late. In other words, please submit on time!

Note that C^1 means "continuously differentiable".

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a C^1 function; we write f in the form $f(x, y_1, y_2)$. Assume that $f(3, -1, 2) = 0$ and

$$f'(3, -1, 2) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(a) Show that there is a function $g\colon B \to \mathbb{R}^2$ defined on an open set B in $\mathbb{R}$ such that $3 \in B$ and such that $g(3) = (-1, 2)$ and

$$f(x, g_1(x), g_2(x)) = 0$$

for $x \in B$.

(b) Find $g'(3)$.

(c) Discuss the problem of solving the equation $f(x, y_1, y_2) = 0$ for an arbitrary pair of the unknowns in terms of the third, near the point $(3, -1, 2)$.

Q2 (10 points)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be C^1, with $f(2, -1) = -1$. Set

$$G(x, y, u) = f(x, y) + u^2,$$

$$H(x, y, u) = ux + 3y^3 + u^3.$$

The equations $G(x, y, u) = 0$ and $H(x, y, u) = 0$ have the solution $(x, y, u) = (2, -1, 1)$.

(a) What conditions on f' ensure that there are C^1 functions $x = g(y)$ and $u = h(y)$ defined on an open set in $\mathbb{R}$ that satisfy both equations, and such that $g(-1) = 2$ and $h(-1) = 1$?

(b) Under the conditions of (a) and assuming that $f'(2, -1) = \begin{pmatrix} 1 & -3 \end{pmatrix}$, find $g'(-1)$ and $h'(-1)$.

Q3 (10 points)

Let $f, g: \mathbb{R}^3 \to \mathbb{R}$ be C^1 functions. "In general", one expects that each of the equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$ represents a "nice" surface in $\mathbb{R}^3$ and that their intersections is a smooth curve. Show that if (x_0, y_0, z_0) satisfies both of these equations, and if $\partial(f, g)/\partial(x, y, z)$ has rank 2 at $p_0 = (x_0, y_0, z_0)$, then near p_0 one can solve these equations for two of x, y, z in terms of the third, thus representing the solution set locally as a parametrized curve.

Note. There is only one reasonable way to interpret the notation $\partial(f, g)/\partial(x, y, z)$.

Q4 (10 points)

Let $f: \mathbb{R}^{k+n} \to \mathbb{R}^n$ be a C^1 function; suppose that $f(a) = 0$ and that $f'(a)$ has rank n. Show that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.