MAT257 Final Assessment Information and Rejected Questions

- The Final Assessment will take place on Tuesday April 20, 9AM-Noon (Toronto time), on Crowdmark (you will get a link by email about one minute before the official starting time). Other than documented accessibility matters, no exceptions!
- Our TAs Peter and Petr will hold extra pre-test office hours, in their usual zoom rooms. Peter on Saturday 9-12 and on Sunday 1-4 at Peter's Zoom (password vchat), and Petr on Monday 9-11 at Petr's Zoom (password vchat).
- I will hold my regular office hours on Tuesday April 13, at 9-10 and 12-1 and then extra hours on Saturday 1-3PM (check again right before), Sunday 9-11, and Monday 1-3, at `http://drorbn.net/vchat`.
- I will be available to answer questions throughout the exam, at my usual office (`http://drorbn.net/vchat`, but I'll add a waiting room). I will also be monitoring my regular email address (drorbn@math.toronto.edu) throughout the exam.
- There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday April 21 at 7PM. I will deal with these situations on a case by case basis.
- Don't let unanswered questions and/or mishaps paralyze you! If you need an answer but for whatever reason you cannot reach me, think hard, come up with what you think is the most reasonable answer/resolution, document as best as you can (for example, by adding a note on your submission), and act following your conclusions.
- Material: *Everything* excluding the material on Maxwell equations, with very light emphasis on the material not covered in the previous term tests,
- Open book(s) and open notes but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "8 of 8".
- You will be required to copy in your handwriting and sign an academic integrity statement and submit it on Crowdmark along with the rest of your exam. If you wish, you may save time by preparing the academic integrity statement in advance as in this sample.
- You will be given an extra 20 minutes at the end of the exam to upload it and to copy/sign the academic integrity statement.
- The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. Please! Life is much better if we don't need to go there.
- To prepare: Do the FA "rejects" available below, but more important: make sure that you understand every single bit of class material so far!
- It is not the assessment I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in 2020/21 are not as we want them.

The following questions were a part of a question pool for the 2020-21 MAT257 Final Assessment, but at the end, they were not included.

1. Suppose that the bounded functions f and g are integrable over some rectangle $R \subset \mathbb{R}^n$. Show that fg, f^2, and g^2 are also integrable over R and that $\int_R fg \leq \left(\int_R f^2\right)^{1/2} \left(\int_R g^2\right)^{1/2}$.

2. Prove that the intersection of finitely many open sets in $\mathbb{R}^n$ is open, and give a counterexample to show that this statement may not be true if the intersection is countably infinite.

3. If A and B are disjoint closed sets in $\mathbb{R}^n$, show that there exists disjoint open subsets C and D of $\mathbb{R}^n$ such that $A \subset C$ and $B \subset D$.

4. Recall that the variation $o(f, t)$ of a function $f\colon \mathbb{R} \to \mathbb{R}$ at a point $t \in \mathbb{R}$ is defined to be

$$\lim_{r \to 0} \sup \{|f(x) - f(y)| \colon x, y \in B_r(t)\}.$$

Prove that if f is monotone on some interval $[a, b]$ and $P = (a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b)$ is a partition of $[a, b]$, then

$$\sum_{i=1}^{n-1} o(f, t_i) \leq |f(b) - f(a)|.$$

5. Show that the function $f(x, y) = |xy|^{1/2}$ is continuous at $a = (0, 0)$ and has both its partial derivatives exist at a, yet it is not differentiable at a.

6. Prove that if a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ has a local max at some point $a \in \mathbb{R}^n$, then $f'(a) = 0$.

7. A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|(x - y) - (f(x) - f(y))| \leq \frac{1}{3}|x - y|$ for every $x, y \in \mathbb{R}^n$. Prove that f is surjective (onto).

8. A function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has the property that for every $k > 0$ we have $|(x-y)-(f(x)-f(y))| \leq \frac{1}{k}|x-y|$ for every $x, y \in \mathbb{R}^n$ whose norm is less than e^{-k}. Prove that f is differentiable at 0 and compute its differential $f'(0)$.

9. A differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}^k$ is said to be "submersive" at 0 if rank $f'(0) = k$. Assume such a function f is submersive at 0 and assume also that $f(0) = 0$, and show that there is a function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ which is defined, differentiable, and invertible near 0, and so that $f(g(x_1, \ldots, x_n)) = (x_1, \ldots, x_k)$. (In other words, every submersive function looks like the standard projection $\mathbb{R}^n \to \mathbb{R}^k$ near 0).

10. (a) A subset $A \subset \mathbb{R}$ is known to have content 0. Is it necessarily true that ∂A also has content 0?

    (b) A subset $B \subset \mathbb{R}$ is known to have measure 0. Is it necessarily true that ∂B also has measure 0?

11. If f is a bounded function defined on a rectangle $R \subset \mathbb{R}^n$ and if supp f (the closure of $\{x \in R \colon f(x) \neq 0\}$) is a set of measure 0, show that f is integrable on R and that $\int_R f = 0$.

12. If f is a bounded non-negative function defined on a rectangle $R \subset \mathbb{R}^n$ and if $\int_R f = 0$, show

    (a) For every $b > 0$, the set $\{x \in R \colon f(x) \geq b\}$ has content 0.

    (b) The set $\{x \in R \colon f(x) > 0\}$ has measure 0.

13. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. Prove that there is a smooth function $h\colon \mathbb{R}^2 \to [0, 1]$ such that $h(x, f(x)) = 1$ for every $x \in \mathbb{R}$, yet always, if $x, y \in \mathbb{R}$ and $|y - f(x)| \geq 1$, then $f(x, y) = 0$.

14. Let $m\colon [0, 1] \to M_{n\times n}(\mathbb{R})$ be a path in the space of $n \times n$ matrices, and suppose that for every $t \in [0, 1]$ the columns of $m(t)$ make a basis of $\mathbb{R}^n$. Show that the bases $m(0)$ and $m(1)$ define the same orientation of $\mathbb{R}^n$.

15. Show that if $F = \sum_i f_i(p)(p, e_i)$ and $G = \sum_i G_i(p)(p, e_i)$ are smooth vector fields on $\mathbb{R}^n$, then there is a third smooth vector field $H = \sum_i h_i(p)(p, e_i)$ on $\mathbb{R}^n$ such that

$$D_F \circ D_G - D_G \circ D_F = D_H,$$

where $D_F\colon \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^n)$ is the operation of directional derivative in the direction of F, which maps smooth functions on $\mathbb{R}^n$ to smooth functions on $\mathbb{R}^n$ (and likewise for D_G and D_H).

16. An exploration problem: a 3-vector on $\mathbb{R}^4_{txyz}$ is a function $F\colon \mathbb{R}^4_{txyx} \to \mathbb{R}^3_{xyz}$. It can be regarded as a time-dependent vector field on $\mathbb{R}^3$, and so it makes sense to write grad, curl, and div in this context, and also $\partial_t = \frac{\partial}{\partial t}$. Of course, you also need to consider "scalar functions" $f\colon \mathbb{R}^4 \to \mathbb{R}$, to talk about grad and div. Can you interpret the sequence

$$\Omega^0(\mathbb{R}^4) \xrightarrow{d} \Omega^1(\mathbb{R}^4) \xrightarrow{d} \Omega^2(\mathbb{R}^4) \xrightarrow{d} \Omega^3(\mathbb{R}^4) \xrightarrow{d} \Omega^4(\mathbb{R}^4)$$

in this language of scalar functions, 3-vectors, grad, curl, div, and ∂_t?

17. Prove that the form $xdydz + ydzdx + zdxdy$ is closed but not exact on the 2-dimensional unit sphere $S^2 \subset \mathbb{R}^3_{xyz}$.

18. ω is a smooth 3-form on $\mathbb{R}^7$, and we know that the integral of ω over every 3-cube in $\mathbb{R}^7$ vanishes. Prove that ω itself vanishes.

19. We will say that a 1-form ω on $\mathbb{R}^n$ is "precise" if its integral over any 1-cube depends only on the boundary of that 1-cube (namely, $\partial c_1 = \partial c_2 \implies \int_{c_1} \omega = \int_{c_2} \omega$. Show that a 1-form ω is precise if and only if it is exact.

20. Suppose M is a k-dimensional manifold in $\mathbb{R}^n$, and suppose F is a smooth vector field on M (so in particular $F(x) \in T_xM$ for every $x \in M$). Show that there is some vector field G on some open set $A \supset M$ (in particular, $G(x) \in T_x\mathbb{R}^n$ for every $x \in A$) such that G restricted to M is F. You may need to use one of the precise definitions of a manifold, and something to make the local go global.

21. A smooth vector field E defined on $\mathbb{R}^3$ is known to satisfy $\operatorname{div} E = 0$ outside of $D^3_{1/2}$, the 3-dimensional closed ball of radius $1/2$ in $\mathbb{R}^3$, and it is also known that $\int_{\partial D^3_1} (E \cdot n) dA = 257$, where everything is taken with "standard conventions": orientations, positive normals, and area forms. Compute $\int_{\partial D^3_2} (E \cdot n) dA$ and $\int_{\partial D^3_1(p)} (E \cdot n) dA$, where $D^3_1(p)$ denotes the closed ball of radius 1 about a point p, and p is a point of ∂D^3_2.

22. A subset B of $\mathbb{R}^3$ is the union two infinite lines positioned as on the figure on the left below, and in addition, oriented loops R_1, R_2, and G_i for $i = 1, 2, 3, 4, 5$ are also given as in the same figure. A vector field F is also given, and it is known to be smooth away from B and to satisfy $\operatorname{curl} F = 0$ on $\mathbb{R}^3 \setminus B$. It is known that $\int_{R_1} (F \cdot T) ds = \pi$ and $\int_{R_2} (F \cdot T) ds = e$. Compute $\int_{G_i} (F \cdot T) ds$ for $i = 1, 2, 3, 4, 5$. *Hint.* You may want to also think about 2D subsets of $\mathbb{R}^3$ that are shaped like masks and/or tubes as in the figure below on the right