Hour 55 MAT257 Analysis II on March 1, 2021: Chains and Integration.
Read Along: Spivak 97-108.
HW15 is due tonight!
TT3 is next week, discussion on Wednesday.
Riddle Along: Can you write the function 1+xy+(xy)^2 as a sum f(x)g(y)+h(x)k(y) of two products of functions that depend on just one variable?

$$1\cdot 1 + x\cdot y + x^2\cdot y^2 + x^3y^3 \qquad \int_c d\omega = \int_{\partial c} \omega$$

$$C_k(A) = \begin{pmatrix} k\text{-chains} \\ \text{in } A\end{pmatrix} = \left\{ \sum_{i=1}^m a_i c_i : \begin{array}{l} a_i \in \mathbb{Z} \\ c_i : I^k \to A \\ \text{cont.}/\boxed{\text{diff'able}}/\text{smooth} \end{array} \right\} \Big/ \begin{array}{l}\text{shopping} \\ \text{list} \\ \text{rules}\end{array}$$

$A \subset \mathbb{R}^n$

An Abelian group / a $\mathbb{Z}$-module

$I^k_{(j,\alpha)} : I^{k-1} \to I^k$ by $(x_1 \ldots x_{k-1}) \mapsto (x_1, \ldots, \overset{j}{\alpha}, x_j \ldots x_{k-1})$

$c : I^k \to A$

$c_{(j,\alpha)} := c \circ I^k_{(j,\alpha)}$ "face" $\quad \partial c := \sum_{j=1}^k \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} c_{(j,\alpha)}$

Thm $\partial^2 = 0$. Extends to k-chains!

$x_1 \ldots x_{k-2} \xrightarrow{\partial} I^k$

$\left(\begin{matrix}\alpha \\ i\end{matrix} \quad \begin{matrix}\beta \\ j\end{matrix} \right)$

Lemma $\left(c_{(i,\alpha)}\right)_{(j,\beta)} = \left(c_{(j+1,\beta)}\right)_{(i,\alpha)}$

when $i \leq j$

Verification: Each side

left: $I^{k-2} \xrightarrow{I_{(j,\beta)}} I^{k-1} \xrightarrow{I_{(i,\alpha)}} I^k \xrightarrow{c} A \subset \mathbb{R}^n$

right: $I^{k-2} \xrightarrow{I_{(i,\alpha)}} I^{k-1} \xrightarrow{I_{(j+1,\beta)}} I^k$

Top composition:

$(x_1 \ldots x_{k-2}) \overset{I_{(j,\beta)}}{\longmapsto} (x_1 \; x_{j-1} \overset{j}{\beta} x_j \; \overset{k-1}{x_{k-2}}) \overset{I_{(i,\alpha)}}{\longmapsto} (x_1 \ldots x_{i-1} \overset{i}{\alpha} \ldots \overset{j+1}{\beta} x_j \ldots x_{k-2})$

$\| \quad \square$

Bottom computation:

$(x_1 \ldots x_{k-2}) \overset{I_{(i,\alpha)}}{\longmapsto} (\quad \overset{i}{\alpha} \quad) \overset{I_{(j+1,\beta)}}{\longmapsto} (\ldots \overset{i}{\alpha} \quad \overset{j+1}{\beta} \ldots)$

where $c : I^{k} \longrightarrow A$ $(k-1)$-cube

$$\partial(\partial c) = \partial\left(\sum_{i=1}^{k} \sum_{\alpha \in 0,1} (-1)^{i+\alpha} \overbrace{C_{(i,\alpha)}}^{(k-1)\text{-cube}}\right)$$

$$= \sum_{i=1}^{k} \sum_{\alpha \in 0,1} \sum_{j=1}^{k-1} \sum_{\beta \in 0,1} \left(C_{(i,\alpha)}\right)_{(j,\beta)} (-1)^{i+j+\alpha+\beta}$$

$$= \sum_{\substack{i \le j \\ i \le k \\ j \le k-1}} \sum_{\alpha,\beta \in \{0,1\}} (-1)^{\overset{j+i-1+\beta+\alpha}{i+j+\alpha+\beta}} \left(C_{(\underbrace{j+1}_{i},\underbrace{\beta}_{\alpha})}\right)_{(\underbrace{i}_{j},\underbrace{\alpha}_{\beta})}$$

$$\begin{matrix} j \le i-1 \\ j \le k \\ i-1 \le k-1 \end{matrix}$$

$$+ \underbrace{\sum_{\substack{i > j \\ i \le k,\, j \le k-1}} \sum_{\alpha,\beta} (-1)^{i+j+\alpha+\beta} \left(C_{(i,\alpha)}\right)_{(j,\beta)}}_{\text{same}}$$

$$= -\sum_{\substack{i > j \\ j \le k-1 \\ i \le k}} \sum_{\alpha,\beta} (-1)^{i+j+\alpha+\beta} \left(C_{(i,\alpha)}\right)_{(j,\beta)} = 0.$$

<u>Integration</u> Integrate a k-form on I^{k}

$F\, dx_1 \wedge \cdots \wedge dx_k$

<u>Def</u>

$$\int_{I^{k}} f\, dx_1 \wedge \cdots \wedge dx_k = \int_{I^{k}} f$$

<u>Examples</u>

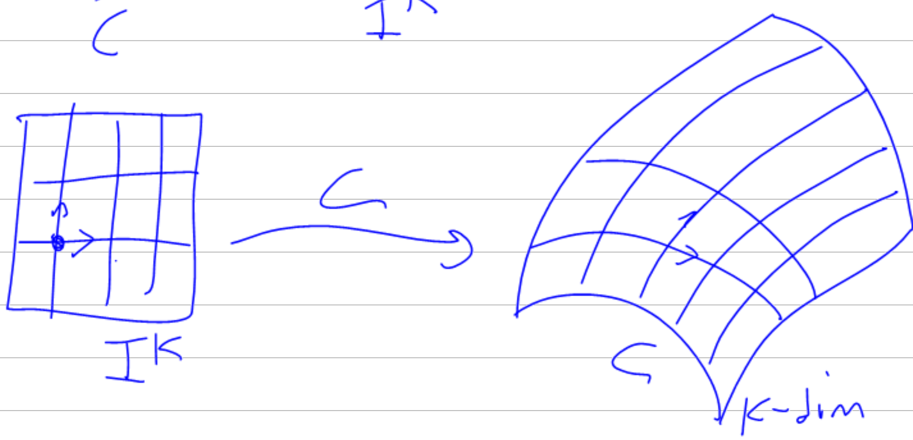
None.

$$\int_{I^0} F_{\_} = f(p) \qquad I^0 = \{p\}$$

$c : I^k \longrightarrow A$ is k-dim cube in $A \subset \mathbb{R}^n$

$\omega \in \Omega^k(A)$

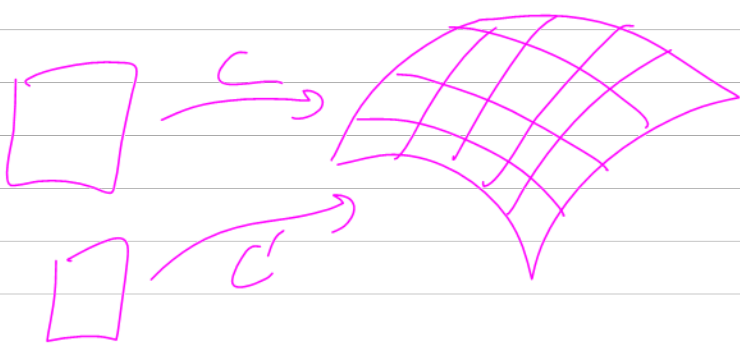
$$\int_c \omega := \int_{I^k} c^*\omega \qquad \begin{array}{c} I^k \xrightarrow{\;c\;} \mathbb{R}^n \\ \qquad\qquad\quad \omega \end{array}$$



$\subset \mathbb{R}^n$

$\omega \in \Omega^k(\mathbb{R}^n)$

$$\int_{\sum a_i c_i} \omega := \sum a_i \int_{c_i} \omega = \sum a_i \int_{I^k} c_i^* \omega$$



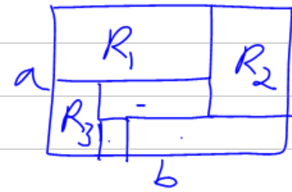
$$\int_{im(c)} \omega$$

MAT257 Term Test 3 Information and Rejected Questions

- The test will take place on Tuesday March 9, 5-7PM (Toronto time), on Crowdmark (you will get a link by email about one minute before the official starting time). Other than documented accessibility matters, no exceptions!
- Our TAs Sebastian and Shuyang will hold extra pre-test office hours, in their usual zoom rooms. Sebastian on Monday 11-2 at Sebastian's Zoom (password vchat), and Shuyang on Friday and on Tuesday at 10:30-11:30 at Shuyang's Zoom (password vchat). These office hours replace some of their regular office hours; so Sebastian will not hold his regular office hours on March 15 and on March 22, and Shuyang will not hold her regular office hours on March 10 and 17.
- I will hold my regular office hours on Tuesday at 9-10 and 12-1, at `http://drorbn.net/vchat`.
- I will be available to answer questions throughout the exam, at my usual office (`http://drorbn.net/vchat`, but I'll add a waiting room). I will also be monitoring my regular email address (drorbn@math.toronto.edu) throughout the exam.
- There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday March 10 at 7PM. I will deal with these situations on a case by case basis.
- Don't let unanswered questions and/or mishaps paralyze you! If you need an answer but for whatever reason you cannot reach me, think hard, come up with what you think is the most reasonable answer/resolution, document as best as you can (for example, by adding a note on your submission), and act following your conclusions.
- Material: Everything up to and including Stokes' Theorem for Chains, with greater emphasis on the material that was not included in Term Test 2 (meaning, starting with the Baby Sard Theorem, and even more so, k-tensors and all that followed).
- Open book(s) and open notes but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- You will be required to copy in your handwriting and sign an academic integrity statement and submit it on Crowdmark along with the rest of your exam. If you wish, you may save time by preparing the academic integrity statement in advance as in this sample.
- You will be given an extra 20 minutes at the end of the exam to upload it and to copy/sign the academic integrity statement.
- The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
- To prepare: Do the TT3 "rejects" available below, but more important: make sure that you understand every single bit of class material so far!
- It is not the test I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in 2020/21 are not as we want them.

Good Luck w/ this (& w/ 357)!

Hour 56 MAT257 Analysis II on March 3, 2021: Integration and Stokes' Theorem.
Read Along: Spivak 97-108.
TT3 is next Tuesday! Info and prep questions at
http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT3-Info.pdf
Riddle Along: A rectangle R has sides that are (both) not integers, and is tiled with rectangles R_i. Show that at least one of the R_i's has sides that are (both) not integer.
A Question for Deep Reflection: Mirrors flip left and right, but not up and down. How can mirrors tell horizontal from vertical?



$a, b \notin \mathbb{Z}$

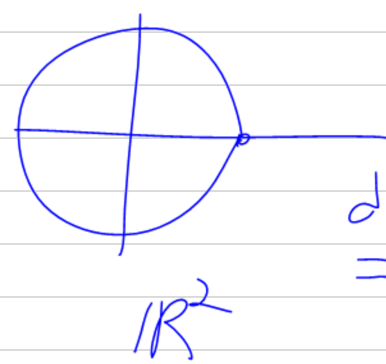
$I^k_{(j,\alpha)} : I^{k-1} \longrightarrow I^k$ by $(x_1 \dots x_{k-1}) \mapsto (x_1, \dots, \underset{j}{\alpha}, x_{j+1} \dots x_k)$

$c_{(j,\alpha)} := c \circ I^k_{(j,\alpha)}$ "face" $\quad \partial c := \sum_{j=1}^{k} \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} c_{(j,\alpha)}$

$\int_{I^k} f\, dx_1 \wedge \dots \wedge dx_k = \int_{I^k} f \qquad \partial^2 = 0$

$A \subset \mathbb{R}^n, \quad c \in C_k(A), \quad \omega \in \Omega^k(A), \quad \int_c \omega := \int_{I^k} c^* \omega$

example 1 $\quad k = 1$



$c : I_t \longrightarrow \mathbb{R}^2$

$c : t \longmapsto \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}$

$d \cos 2\pi t = (-\sin 2\pi t) \cdot 2\pi\, dt$

$\Omega^1(\mathbb{R}^2) \ni \omega = \frac{-y\, dx}{x^2+y^2} + \frac{x\, dy}{x^2+y^2}$

$\int_c \omega = \int_I c^* \omega = \int_I \frac{+\sin^2(2\pi t) \cdot 2\pi\, dt}{1} + \frac{\cos^2(2\pi t)\, 2\pi\, dt}{1} = 2\pi \int_I 1 \cdot dt$

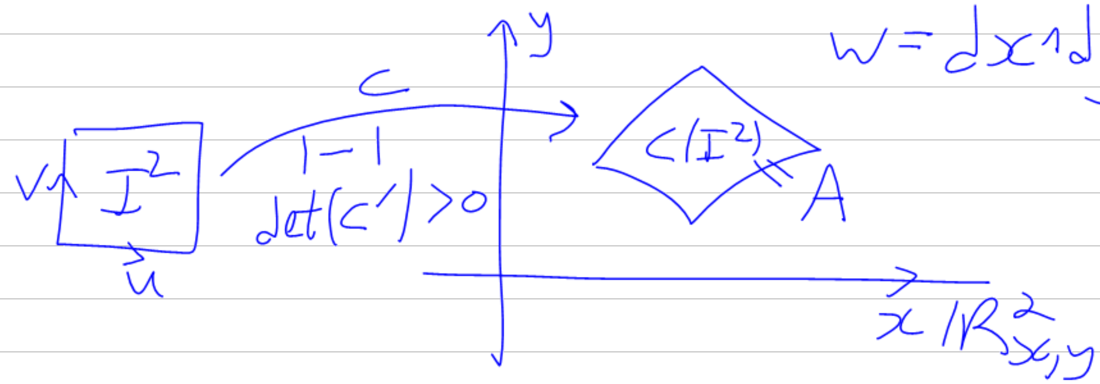
$= 2\pi$

Aside Once we'll know Stokes', we will know that ω is not exact on all of $\mathbb{R}^2$.

$2\pi = \int_c \omega \underset{\text{suppose } \omega = d\lambda}{\overset{\text{by neg,}}{=\!=\!=}} \int_c d\lambda \overset{\text{Stokes'}}{=} \int_{\partial c} \lambda = \int_0 \lambda = 0 \quad \Rightarrow\Leftarrow$

Example

$w = dx \wedge dy$



$$\int_c w = \int_{I^2} c^*(w) = \int_{I^2} (\det c')\, du \wedge dv = \int_{I^2} |\underbrace{\det c'}_{g}| \cdot \underbrace{1}_{f\circ g}$$

$$\overset{\text{cov}}{=} \int_A 1 = \mathrm{vol}(A)$$

Ex check that under same cond,

$$\int_c f\, dx \wedge dy = \int_A f$$

$\boxed{I^2} \xrightarrow[c]{g} \square \xrightarrow[1]{f} \mathbb{R}$

$$\int_{g(I^2)} f = \int_{I^2} (f\circ g) |\det g'|$$

$$\int_A 1 = \int_{I^2} 1 \cdot |\det c'|$$

Great omission!

$I^k \xrightarrow{c} A \xrightarrow{g} B \to$

$A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$

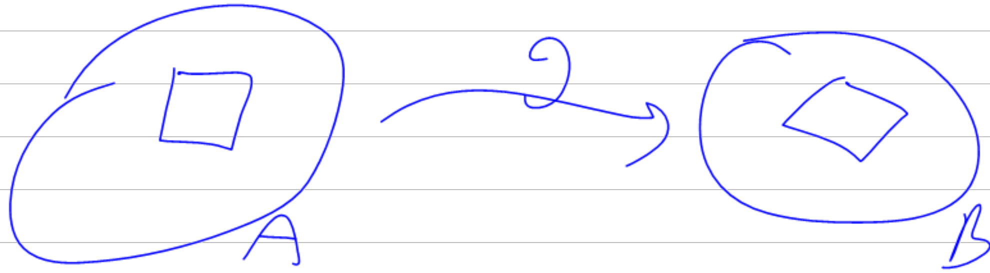
$$C_k(A) \xrightarrow{g_*} C_k(B)$$

$$g_*\left(\sum a_i c_i\right) = \sum a_i (g \circ c_i)$$

$a_i \in \mathbb{Z}$

$c_i : I^k \to A$

Claim! $g_*(\partial c) = \partial(g_* c)$

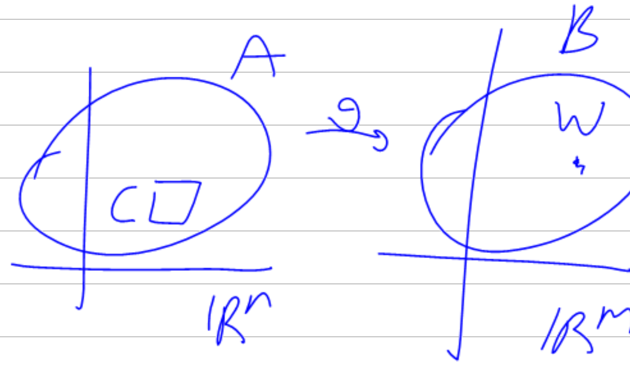


PF Follow the defs. $\partial C = \sum \omega \; C \circ I^k_{(j,\alpha)}$

It boils down to $g_* C = g \circ C$

"composing on right commutes w/ composing on left"

Claim If $g : \underset{\mathbb{R}^n}{A} \to \underset{\mathbb{R}^m}{B}$



& $C \in C_k(A)$, $w \in \Omega^k(B)$,

then $\displaystyle\int_C g^* w = \int_{g_* C} w$

PF Follow the defs. WLOG, $C : I^k \to A$.

$$\int_{I^k} C^*(g^* w) \qquad\qquad \int_{I_k} (g_* C)^* w = \int_{I_k} (g \circ C)^* w$$

pullbacks are contravariant

Thm $c \in C_k(A)$, $\omega \in \Omega^{k-1}(A)$

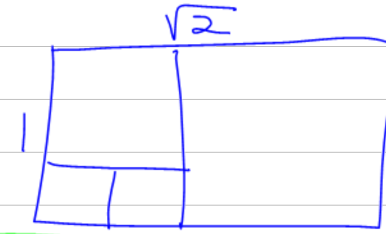
Then
$$\int_c d\omega = \int_{\partial c} \omega$$

In $\mathbb{R}^3$

$$\Omega^0 \xrightarrow[\text{grad}]{d} \Omega^1 \xrightarrow[\text{curl}]{d} \Omega^2 \xrightarrow[\text{div}]{d} \Omega^3$$

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = i^2 = -1$$

Hour 57 MAT257 Analysis II on March 5, 2021: Stokes' Theorem.
Read Along: Spivak 97-108.
TT3 is next Tuesday! Info and prep questions at
http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT3-Info.pdf
Riddle Along: Can you partition a rectangle exactly one of whose sides is irrational into finitely many squares?
A Question for Deep Reflection: Mirrors flip left and right, but not up and down. How can mirrors tell horizontal from vertical?



$$(x_1 \ldots x_{k-1}) \xmapsto{I^k_{(j,\alpha)}} (x_1, \ldots, \underset{j}{\alpha}, x_j \ldots x_k) \qquad \partial c = \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha} c \circ I^k_{(j,\alpha)}$$

$$c \in C_k(A\subset\mathbb{R}^n),\ \omega\in\Omega^k(A) \quad \int_c \omega := \int_{I^k} \underset{f\,dx_1\ldots dx_k}{\underset{\parallel}{c^*\omega}} := \int_{I^k} f$$

Thm $c \in C_k(A\subset\mathbb{R}^n),\ \omega\in\Omega^{k-1}(A) \Rightarrow \int_c d\omega = \int_{\partial c}\omega$

PF WLOG, $c: I^k \longrightarrow A$ (diff'ble)

$$c: \underbrace{I^k \xrightarrow{id} I^k}_{\substack{\text{enough to prove} \\ \text{Stokes' for } id:\, I^k \to I^k}} \xrightarrow{c} A$$

Indeed

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(\overbrace{c^*\omega}^{\omega'}) \overset{\substack{\text{use} \\ \text{Stokes'} \\ \text{for } \omega' \\ \&\ I^k}}{=} \int_{\partial I^k} c^*\omega$$

$$= \int_{c_*(\partial I^k)} \omega = \int_{\partial(c_* I_k)} \omega = \int_{\partial c} \omega$$

Claim Stokes' holds for $id: I^k \longrightarrow I^k$

Now $\omega \in \Omega^{k-1}(I^k)$

So $\omega = \sum_{j=1}^{k} F_j \, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k$

wlog, $\omega = F \, dx_{no\,j}$

$$lhs = \int_{I^k} d(F dx_{no\,j}) = \int_{I^k} (dF) \wedge dx_{no\,j}$$

$$dF = \left(\sum_i \frac{\partial F}{\partial x_i} dx_i\right)$$

$$= \int_{I^k} \frac{\partial F}{\partial x_j} \underbrace{dx_j \wedge dx_{no\,j}}_{(-1)^{j-1} dx_{tot}} = (-1)^{j-1} \int_{I^k} \frac{\partial F}{\partial x_j}$$

$$\xrightarrow[\text{FTC}]{\text{Fubini}} (-1)^{j-1} \int_{I^{k-1}} F(x_j = 1) - F(x_j = 0)$$

$$= (-1)^{j-1} \int_{I^k_{(j,1)}} F dx_{no\,j} - (-1)^{j-1} \int_{I^k_{(j,0)}} F dx_{no\,j}$$

rhs $= \int F dx_{no\,j}$

$$\partial I^k = \sum_{i,\alpha} (-1)^{i+\alpha} I^k_{(i,\alpha)}$$

$$= \sum_{i,\alpha} (-1)^{i+\alpha} \int_{I^k_{(i,\alpha)} : (y_1 \dots y_{k-1}) \mapsto (y_1 \dots \overset{i}{\alpha} \, y_i \dots y_k)} F \, dx_{noj} =$$

$$= \sum_{i,\alpha} (-1)^{i+\alpha} \int_{I^{k-1}_{y_1 \dots y_{k-1}}} I^{k\,*}_{(i,\alpha)} F \, dx_{noj} = \#$$

$$\left(I^k_{(i,\alpha)}\right)^* \left(dx_1 \dots \widehat{dx_j} \dots dx_k \right)$$

$$= \underbrace{d\left(I^{k\,*}_{(i,\alpha)} x_1\right) \wedge \dots d \widehat{I^{k\,*}_{(i,\alpha)} x_j} \wedge \dots d\left(I^{k\,*}_{(i,\alpha)} x_k\right)}$$

if $i \neq j$ this product contains

$$d\Big(\underbrace{I^{k\,*}_{(i,\alpha)}\, x_i}_{\alpha}\Big) = 0$$

If $i = j$

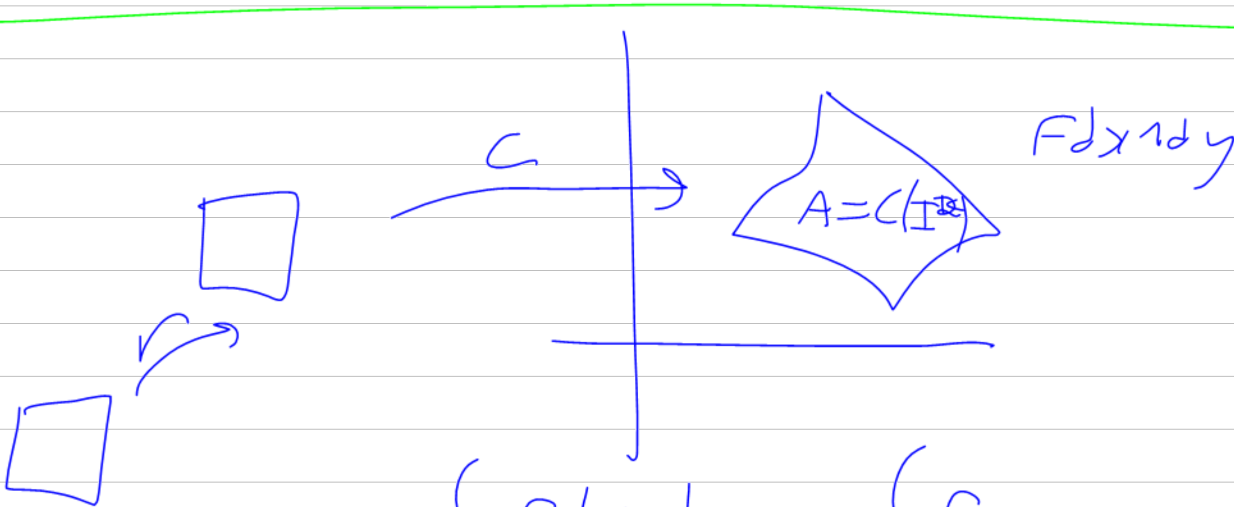
$dy_1 \wedge \dots dy_{k-1}$

$$\# = \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} \int_{I^{k-1}_{y_1 \dots y_{k-1}}} I^{k\,*}_{(j,\alpha)} F \, dx_{noj}$$

$$= \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} \int_{I^{k-1}} F(y_1 \dots \overset{j}{\alpha} \dots y_{k-1}) \, dy_1 \wedge \dots \wedge dy_{k-1}$$

$$= \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} \int_{I^{k-1}} F(y_1 \ldots \overset{j}{\alpha} \ldots y_{k-1}) = \text{lhs}$$

$\square$



$$\int_c F\,dx \wedge dy = \int_A F$$

Exercise Given $c : I^k \longrightarrow A \subset \mathbb{R}^n$ & $w \in \Omega^k(A)$ & $r : I^k \longrightarrow I^k$, 1-1, onto, $\det r' > 0$ then

$$\int_c w = \int_{c \circ r} w$$