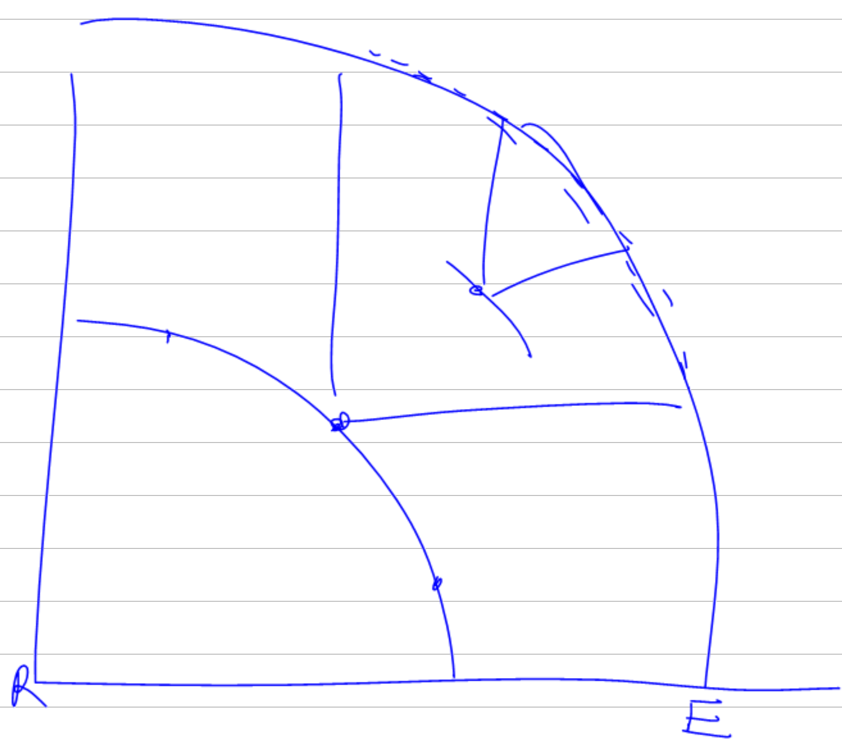
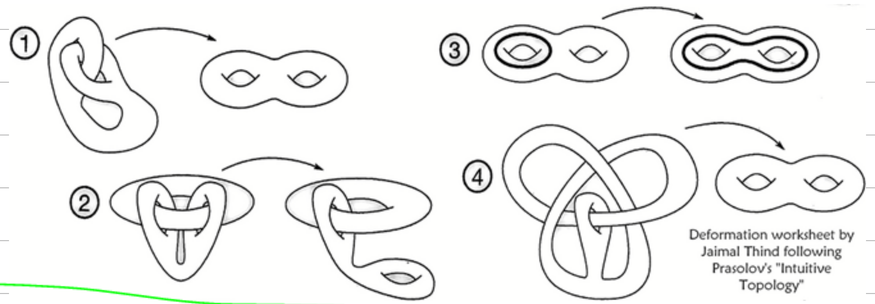

R
E

MAT257 Analysis II on February 22, 2021: d.
Read Along: Spivak 86-92.
Reminder - TA Office Hours: Sebastian Monday 11-12, Shuyang Wednesday 3-4; a great resource!
HW14 due tonight, HW15 on web by midnight!
Riddle Along. On Z x Z, a visible roach R starts at (0, 0) and once a minute jumps to the northeast, up to a distance of 10. Meanwhile, an exterminator E can poison one grid point per minute, away from R. Can E trap R?

$\Omega^k(\mathbb{R}^n)$



$d: \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^n dx_i \wedge \frac{\partial \omega}{\partial x_i}$ $\quad \omega = \sum a_I dx_I \quad \frac{\partial \omega}{\partial x_i} = \sum \frac{\partial a_I}{\partial x_i} dx_I$

Properties 0. At $k=0$ $\quad d_{new} f = d_{old} f$ $\square$

1. Linear $d(\omega_1 + \omega_2) = \dots \quad d(\alpha\omega) = \dots$ $\square$

2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$ Leibnitz law $\square$

$(fg)' = f'g + fg'$

3. $d(g^* \omega) = g^*(d\omega)$

4. $\Omega^k \xrightarrow{d^k} \Omega^{k+1} \xrightarrow{d^{k+1}} \Omega^{k+2}$ $\quad d \circ d = 0$, $d^2 = 0$

$$\begin{tikzcd} \Omega^k \arrow[r, "d^k"] \arrow[rr, bend right, "0"'] & \Omega^{k+1} \arrow[r, "d^{k+1}"] & \Omega^{k+2} \end{tikzcd}$$

PF OF 4 $\quad \omega \in \Omega^k(\mathbb{R}^n)$

$$d^2\omega = d(d\omega) = d\left(\sum_i dx_i \wedge \frac{\partial \omega}{\partial x_i}\right)$$

$$= \sum_j dx_j \wedge \frac{\partial}{\partial x_j}\left(\sum_i dx_i \frac{\partial \omega}{\partial x_i}\right)$$

$$\overset{①}{=} \underbrace{\sum_{i,j} \underbrace{dx_j \wedge dx_i}_{\text{anti-symmetric in } i \,\&\, j} \wedge \underbrace{\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}}_{\text{symmetric in } i,j} \omega}_{\text{anti-symmetric}}$$

$f: \mathbb{R}^n \to \mathbb{R}$ smooth

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$= \sum_{j,i} \left(-dx_i \wedge dx_i\right) \wedge \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} w \qquad i \leftrightarrow j$$

$$= \sum_{i,j} - dx_j \wedge dx_i \wedge \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} w \qquad (2)$$

$$= 0.$$

Pf 3

$$\mathbb{R}^n_{x_1 \ldots x_n} \xrightarrow{g} \mathbb{R}^m_{y_1 \ldots y_m}$$

$$w = \sum_{I \in \mathcal{I}^k_n} f_I \, dy_I$$

$$g^*(dw) = d(g^* w):$$

$$0 = \cdots dy_{i_1} \wedge \cdots \wedge d dy_{i_j} \wedge \cdots dy_{i_k} = d\left(dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_k}\right)$$

$$g^*(dw) = g^*\left(\sum_I d\left(f_I \wedge dy_I\right)\right)$$

$$= g^*\left(\sum_I df_I \wedge dy_I + f_I \underbrace{\cancel{d(dy_I)}}_{0}\right)$$

$$= \sum_I g^*\left(df_I \wedge dy_I\right)$$

$$= \sum_I \left(g^*(df_I)\right) \wedge g^*(dy_I)$$

$$= \sum_I d(g^* f_I) \wedge g^*(dy_I)$$

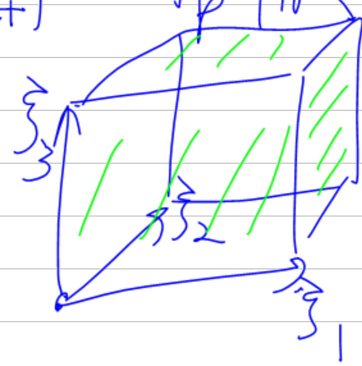
On the other hand

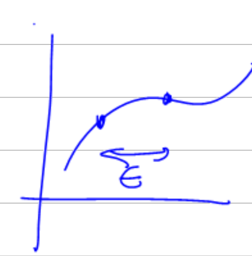
$$d\left(g^*\left(\sum_I f_I \, dy_I\right)\right) = \sum_I d\left(g^*(f_I \wedge dy_I)\right)$$

$$= \sum_I d\left((g^* f_I) \wedge g^*(dy_I)\right)$$

$$= \sum_I \left((dg^* f_I) \wedge g^*(dy_I) + \underbrace{\cancel{dg^*(dy_I)}}_{0}\right) \qquad \square$$

$w \in \Omega^k(\mathbb{R}^n)$ $\quad \xi_1 \dots \xi_{k+1} \in T_p(\mathbb{R}^n)$

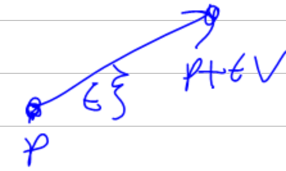
$P(\xi_1 \dots \xi_{k+1}) =$ 



Thm

$$(dw)(\xi_1 \dots \xi_{k+1}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \int_{\partial P(\epsilon \xi_1, \epsilon \xi_2 \dots \epsilon \xi_{k+1})} w$$

$k=0$: $\quad w = f$

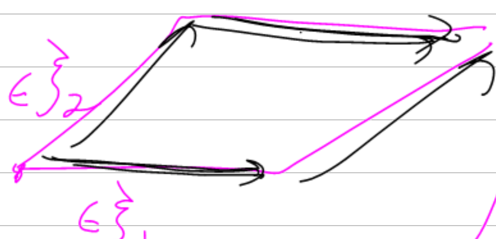


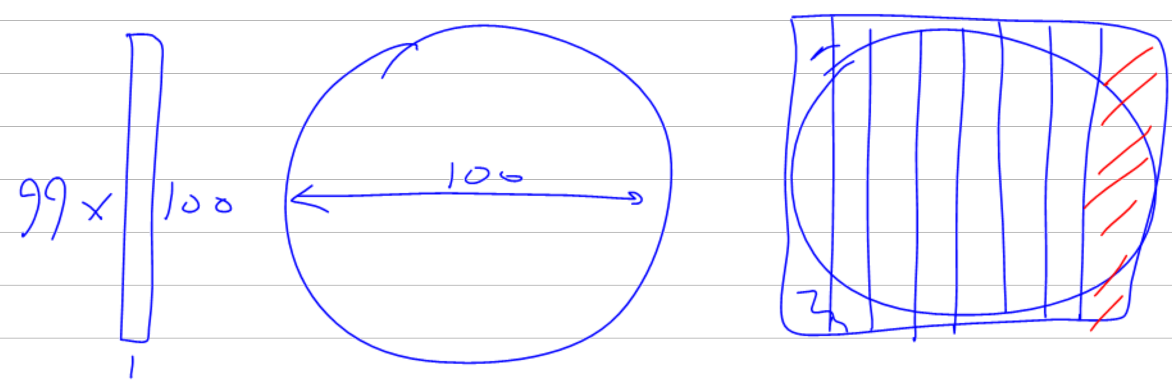
$$(df)(\xi) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\overbrace{f(p+\epsilon V) - f(p)}^{\int_{\partial P} w}\right)$$

$$= D_\xi f \qquad !$$

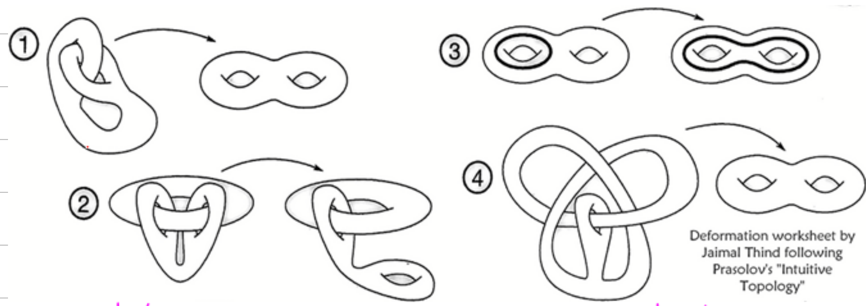
$$\int_C dw = \int_{\partial C} w$$

$k=1 \quad \xi_1 \quad \xi_2$

$w\Big(\partial$  $\Big)$



MAT257 Analysis II on February 24, 2021: The geometric meaning of d.
Read Along: Spivak 86-92.
Riddle Along. Can you cover a diameter 100 disk with 99 (possibly overlapping) 100 x 1 bands?



$d : \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

$df(\xi) = D_\xi f$, linear, (super)-Leibnitz, $d(\omega \wedge \eta) = (d\omega)\wedge\eta + (-1)^{\deg \omega}\omega\wedge d\eta$

compatible w/ pullbacks, $d^2 = d \circ d = 0$

Thm $(d\omega)(\xi_1, \dots, \xi_{k+1}) =$ where $\xi_i = (p, v_i)$

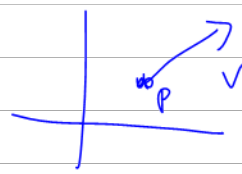
$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \sum_{j=1}^{k+1} (-1)^{j-1} \Big[\omega(p+\epsilon v_j)(\epsilon v_1, \dots \widehat{\epsilon v_j}, \dots \epsilon v_{k+1})$$
$$- \omega(p)(\epsilon v_1, \dots \widehat{\epsilon v_j}, \dots, \epsilon v_{k+1}) \Big]$$

$(1 \dots 8) \quad (1..4\,6..8) \sim (1 \dots \hat{5} \dots 8)$

@$k=0$: $\omega \sim f \qquad \xi = (p, v)$

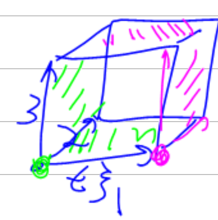
$$df(\xi) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big[f(p+\epsilon v) - f(p) \Big] = D_\xi f$$

diff is small rel to ϵ^{k+1}



$$\epsilon^{k+1}(d\omega)(\xi_1 \dots \xi_{k+1}) \sim \sum_{j=1}^{k} (-1)^{j-1} \Big[\omega(p+\epsilon v_j)(\epsilon v_1\ \widehat{\epsilon v_j}\ \epsilon v_{k+1}) - \omega(p)(\epsilon v_1\ \widehat{\epsilon v_j}\ \epsilon v_{k+1}) \Big]$$

$d\omega(\epsilon\xi_1 \dots \epsilon\xi_{k+1}) \sim (\ \ \ \ \ \ )$

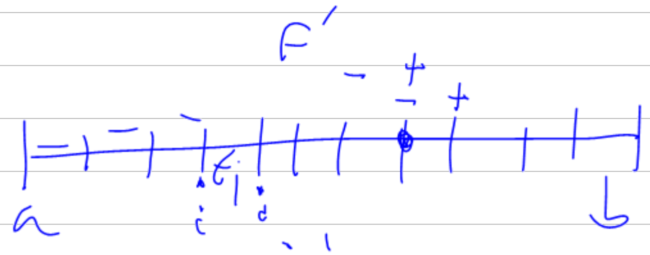


$P = P(\epsilon\xi_1 \dots \epsilon\xi_{k+1})$ = the parallelopiped defined by $(\epsilon\xi_i)$'s.

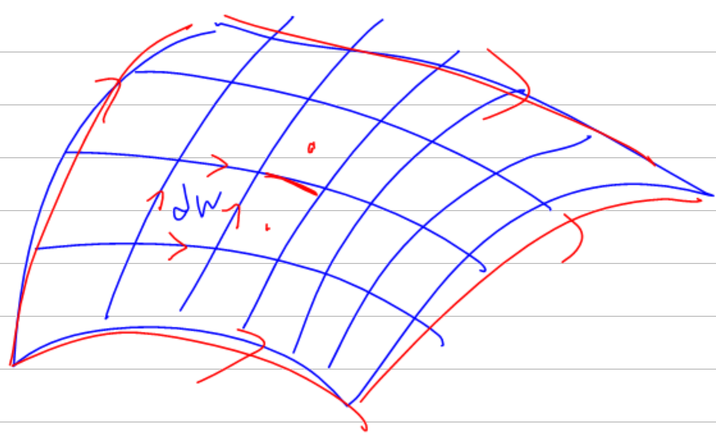
$$\int_P d\omega = \int_{\partial P} \omega$$ "infinitesimal version of Stokes' thm"

$$\int_{[a,b]} F' = \int_{\partial[a,b]} F \qquad F\Big|_a^b$$

$$\epsilon F'(\xi) \sim F(p+\epsilon v) - F(p)$$



$\sim F(b) - F(a)$



Thm $(d\omega)(\xi_1, \dots \xi_{k+1}) =$ where $\xi_i = (p, v_i)$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big[\omega(p+\epsilon v_i)(\epsilon v_1, \dots \widehat{\epsilon v_i}, \dots \epsilon v_{k+1}) - \omega(p)(\epsilon v_1, \dots \widehat{\epsilon v_i}, \dots, \epsilon v_{k+1}) \Big]$$

PF $\omega = \sum \underbrace{f_I}\, \underbrace{dx_I}$ It's enough to prove

thm for forms of the form $f \cdot \lambda$

$f: \mathbb{R}^n \to \mathbb{R}$ is a smooth fnctn,

& λ is a constant diff form, meaning

a form whose coeff functions are constant.

lhs: $d\omega = d(f \wedge \lambda) = df \wedge \lambda + f \wedge \cancel{d\lambda} = (df) \wedge \lambda$

$$(d\omega)(\xi_1 \dots \xi_{k+1}) = ((df) \wedge \lambda)(\xi_1 \dots \xi_{k+1}) =$$

$$\sum_{i=1}^{k+1} (-1)^{i-1} (df)(\xi_i)\, \lambda(\xi_1 \dots \hat{\xi_i} \dots \xi_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1} (D_{\xi_i} f) \lambda(\xi_1 \dots \hat{\xi_i} \dots \xi_{k+1})$$

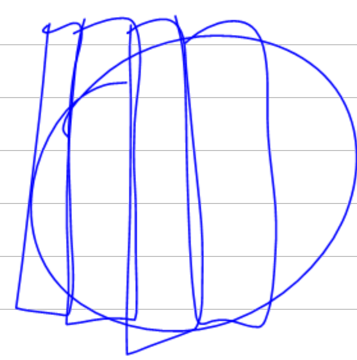
$$\boxed{\omega = f \wedge \lambda}$$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \sum (-1)^{i-1} \begin{bmatrix} \omega(p+\epsilon v_i)(\epsilon v_1 \dots \widehat{\epsilon v_i} \dots \epsilon v_{k+1}) \\ - \omega(p)(\dots) \end{bmatrix}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum (-1)^{i-1} \left[\begin{pmatrix} f(p+\epsilon v_i) \\ - f(p) \end{pmatrix} \left(\lambda(\xi_1 \dots \hat{\xi_i} \dots \xi_{k+1}) \right) \right]$$

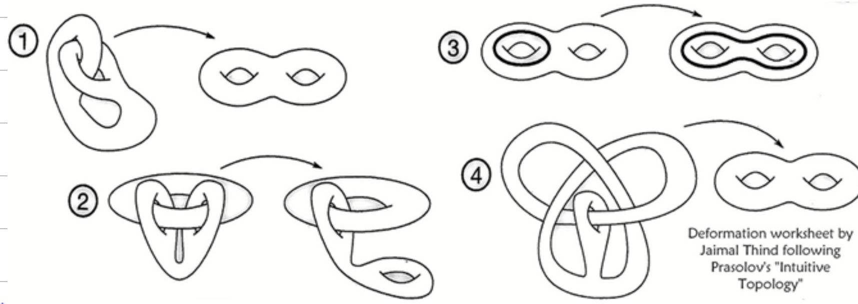
$$= \sum (-1)^{j-1} (D_{\xi_j} F)(\lambda(\xi_1 \ldots \hat{\xi}_j \ldots \xi_{k+1})) = \text{lhs} \quad \square.$$

$$\int_C d\omega = \int_{\partial C} \omega$$



Hint: related bread in bread cutting machine riddle

MAT257 Analysis II on February 26, 2021: Chains.
Read Along: Spivak 92-93, (skip Poincare), 97-108.
Riddle Along: A function f:Z^3→R is superharmonic if for every x, f(x) is greater than or equal to the average of f on the 6 points adjacent to x. Is there a non-negative, non-constant, superharmonic function on Z^3?



$d: \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

$df(\xi) = D_\xi f$, linear, (super)-Leibnitz, compatible w/ pullbacks, $d^2 = d \circ d = 0$, satisfies "infinitesimal $\int_P d\omega = \int_{\partial P} \omega$

Aside on $d^2 = 0$: $A \subset \mathbb{R}^n$ is open

$$\begin{tikzcd} \Omega^{k-1}(A) \arrow[r, "d^{k-1}"] \arrow[rr, bend right, "0"'] & \Omega^k(A) \arrow[r, "d^k"] & \Omega^{k+1}(A) \end{tikzcd}$$

$\Rightarrow \operatorname{im} d \subset \ker d$ $\quad [\operatorname{im} d^{k-1} \subset \ker d^k]$

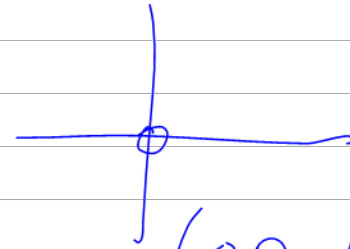
$\operatorname{im} d = \{\text{exact forms}\}$ $\quad$ ω is exact, if $\exists \lambda$ s.t. $\omega = d\lambda$

$\ker d = \{\text{closed forms}\}$ $\quad$ ω is closed if $d\omega = 0$.

We know that every exact form is closed.

<u>Poincaré's Lemma</u>: On $\overset{A}{\underset{\|}{\mathbb{R}^n}}$ every closed form is exact, $\operatorname{im} d = \ker d$.

$A = \mathbb{R}^2 \setminus \{0\}$



$$\omega = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$$ (on HW assignment)

is closed (easy though tedious) $\in \Omega^1(A)$

but not exact (later)

<u>Def</u> Given $A \subset \mathbb{R}^n$, a "singular k-cube in A" is a $\underset{\text{smooth}}{\overset{\text{cont}}{\boxed{\text{diffentable}}}}$ map $c : [0,1]^k \longrightarrow A$

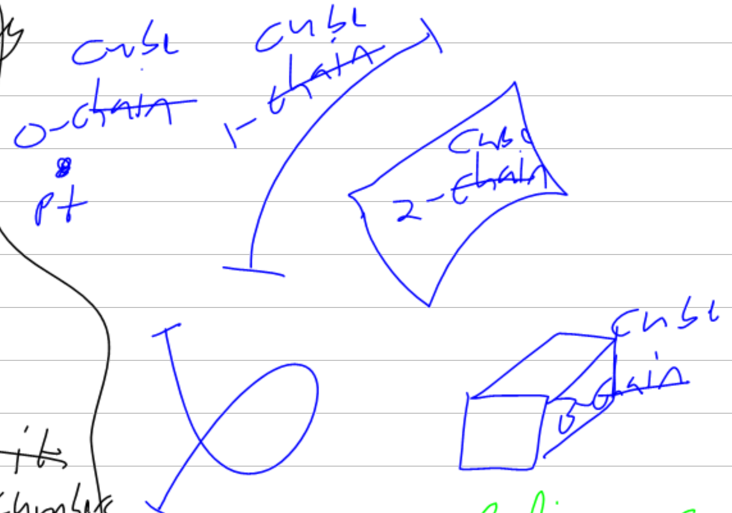
Formal lin-comb: shopping/inventory list

7 banana + 3 apples − tomato

$(7\,\text{ban} + 3\,\text{apples}) + (2\,\text{ban} + 1\,\text{pota})$
$= \underbrace{(9b + 3a + 1p)}_{\text{order immaterial}}$

0 bannanas = scratch it

"0" = empty list = 0 cucumbers



<u>Def</u> $A \subset \mathbb{R}^n$

$$C_k(A) = \left\{ \begin{array}{c} \text{all k-chains} \\ \text{in } A \end{array} \right\} = \Big\{ \overbrace{\sum_{i=1}^{m} a_i c_i}^{\text{Formal linear combinations}} : \begin{array}{l} a_i \in \mathbb{Z} \\ c_i \text{ is} \\ \text{a k-cube} \end{array} \Big\}$$

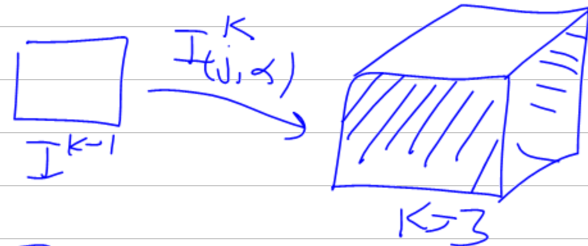
order immaterial
drop 0's
"0" $= \sum_{i=1}$ —— $= 0$ $\square$

<u>Prop</u>: is an Abelian group: has +, inverses, 0, $\overset{\mathbb{Z}}{\underset{\downarrow}{}}$ a.

Goal:

$$C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial} C_{k-2}$$

<u>Def</u> $(I^k = [0,1]^k)$



$I^k_{(j,\alpha)} : I^{k-1} \longrightarrow I^k$

given by $\left[\begin{array}{l} \text{where } 1 \le j \le k \\ \& \ \alpha \text{ is } 0 \text{ or } 1 \end{array}\right]$

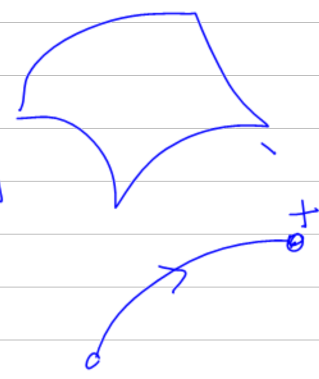
$$(x_1 \ldots x_{k-1}) \longmapsto (x_1 \ldots x_{j-1}\ \alpha\ x_j \ldots x_{k-1})$$

Suppose C is a cube in A: $C: I^k \to A$

For $1 \le j \le k$ & $\alpha \in \{0,1\}$, set

$$C_{(j,\alpha)} = C \circ I_{(j,\alpha)} : I^{k-1} \to A$$

is a $k-1$ cube in A.

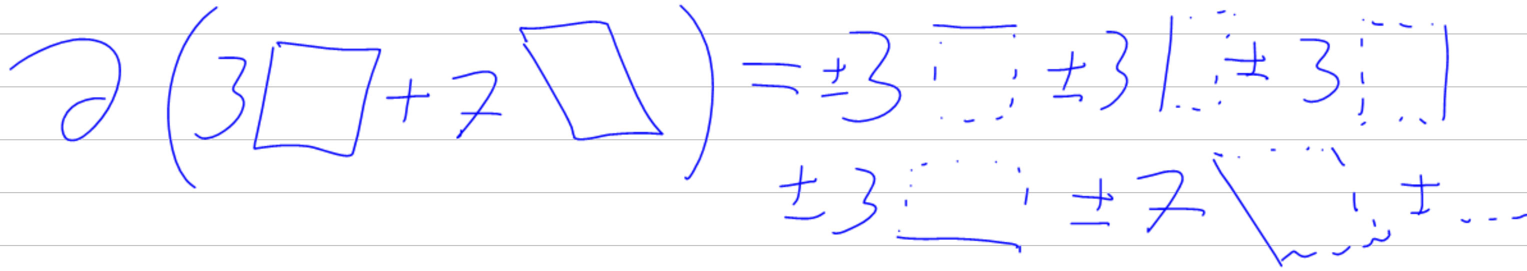


<u>Def</u> If $C: I^k \to A$ is a cube,

$$\partial C = \sum_{j=1}^{k} \sum_{\alpha=0}^{1} (-1)^{j+\alpha} C_{(j,\alpha)} \in C_{k-1}(A)$$

extend to formal lin comb by insisting on linearity:

$$\partial\left(\sum_{i=1}^{m} a_i C_i\right) := \sum_{i=1}^{m} a_i \partial C_i \in C_{k-1}$$



$$C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1}$$

Lemma $\partial^2 = 0$ $\quad \partial_k \circ \partial_{k+1} = 0$