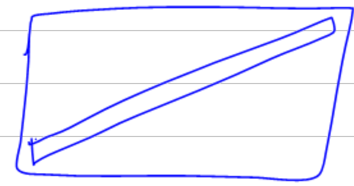
MAT257 Analysis II on October 14, 2020: The Inverse Function Theorem.
Read Along: Spivak 34-40.
HW5 is on web. HW4 is due by 11:59PM.
Riddle Along: Can you fit a box of dimensions a'xb'xc' inside a box of dimensions axbxc, if a'+b'+c'>a+b+c?

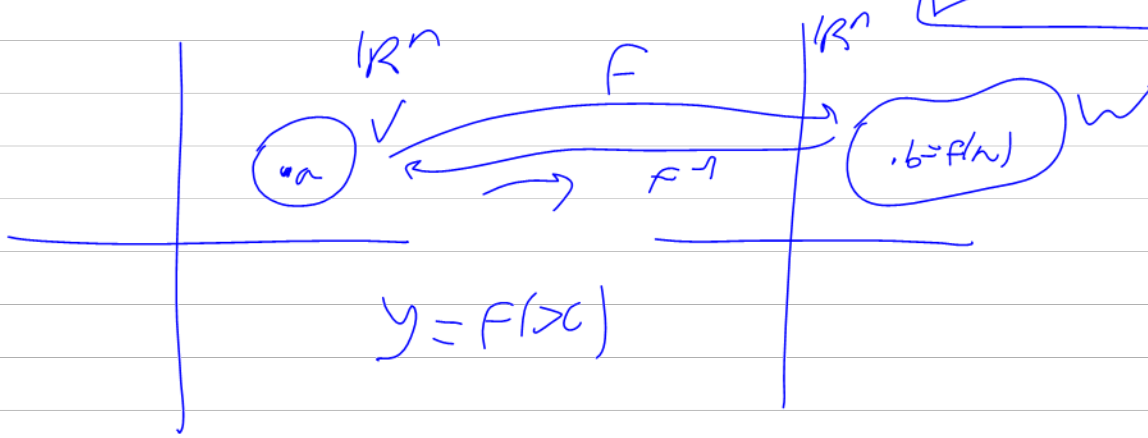
Best with video on but sound off.

**Thm** (The Inverse Function Theorem, IFT)

$F:\mathbb{R}^n\to\mathbb{R}^n$ cont. diffable in an open set $A$ containing $a\in\mathbb{R}^n$, $F'(a)$ invertible $\Rightarrow$ $\exists$ open $V\ni a$, open $W\ni b=F(a)$ s.t. $F|_V : V\to W$ is invertible with $F^{-1}:=(F|_V)^{-1}$ is cont., diffable, and with

$$* \quad (F^{-1})'(y)=\left[F'(F^{-1}(y))\right]^{-1}$$






Pf

1. Assuming $F^{-1}$ exists and is diffable, the formula for $(F^{-1})'$ is easy:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "F"] & \mathbb{R}^n \arrow[r, "F^{-1}"] & \mathbb{R}^n \end{tikzcd}$$

$x$ ~~a~~ $\in V$ $\qquad$ ~~b~~ $\in W$, $y=F(x)$ $\qquad$ $V$

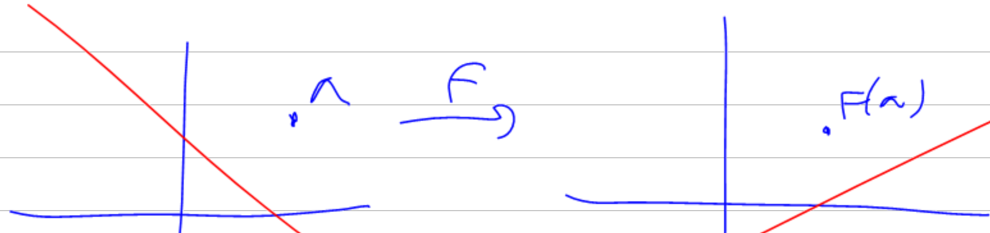
$I$

$$I'(\cancel{a}\,x)=(F^{-1})'(F(\cancel{a}\,x))\circ\underbrace{F'(\cancel{a}\,x)}_{\text{invertible}}$$

$\quad\|$

$\quad I$

$$\text{So } (F^{-1})'(\cancel{b}\,y)=\left(F'(F^{-1}(\cancel{b}\,y))\right)^{-1}$$




~~WLOG $a = f(a) = 0$.~~

~~Define~~

$$\cancel{\bar F(x) = f(a+x) - f(a)}$$

$$\cancel{\bar F(0) = 0}$$

$\lambda = f'(a)$


WLOG, $f'(a) = I$

Define $\bar F(x) = \lambda^{-1}(f(x))$ where $\lambda = f'(a)$. Then, $\bar F$ is cont & diff'ble near $a$, and

$$(\bar F')(x) = \lambda^{-1} \cdot f'(x) \quad \text{so}$$

$$(\bar F')(a) = \lambda^{-1} f'(a) = \lambda^{-1} \cdot \lambda = I$$

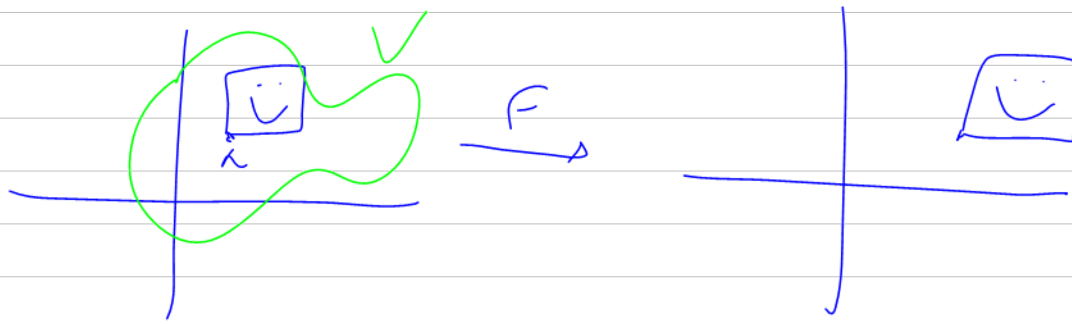
If IFT was true for functions whose differential is $I$, then it's true for $\bar F$, so $\exists (\bar F)^{-1}$. In the B case, define

$F^{-1}(x) = \bar{F}^{-1}(\lambda^{-1}x)$ I claim that $F^{-1}$ is indeed the inverse of $F$.

Indeed,

$$F^{-1}(F(x)) = \bar{F}^{-1}(\lambda^{-1}F(x)) = \bar{F}^{-1}(\bar{F}(x)) = x.$$

From now on assume $F'(a) = I$




Consider $g(x) = F(x) - x$

$$g'(a) = I - I = 0$$

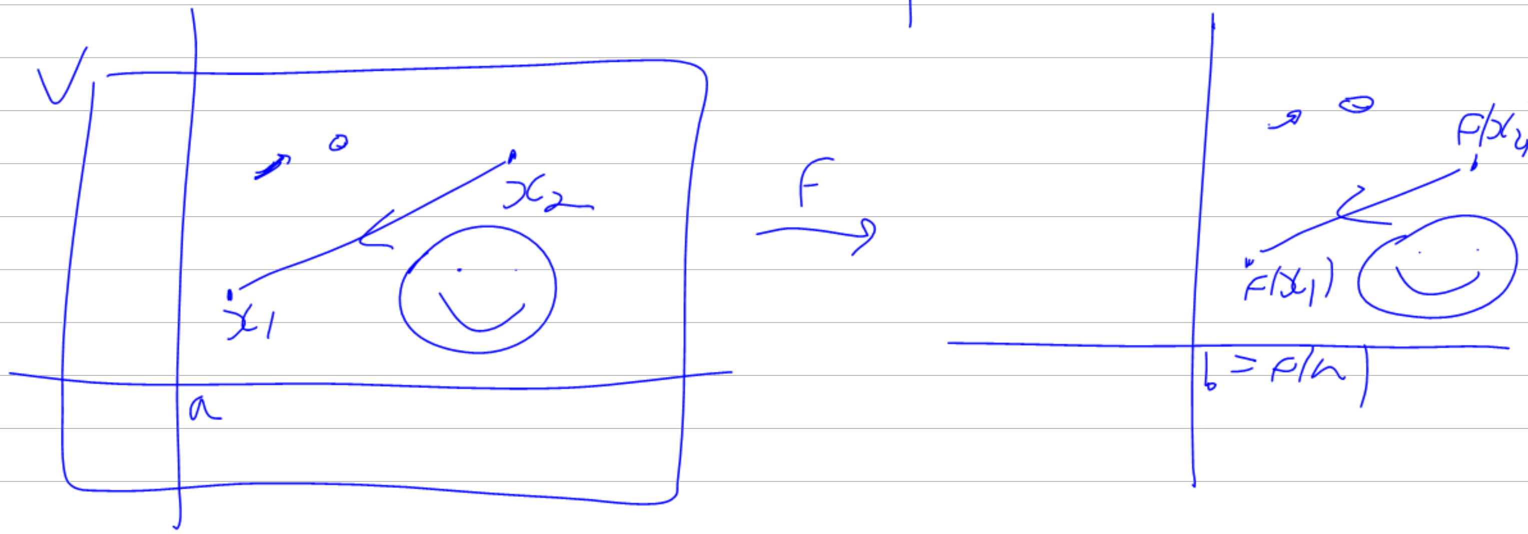
Choose an open rect. $V$ containing $a$ s.t. on $V$

$$\left|D_i g_j(x)\right| \leq \frac{1}{257 \cdot n^2} \quad x \in V$$

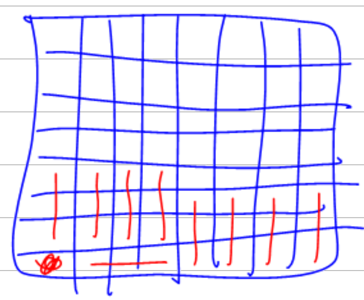
Then by Aside/Lemma, for any $x_1$ & $x_2$ in $V$,

$$\left|g(x_1) - g(x_2)\right| < \frac{1}{257}\left|x_1 - x_2\right|$$

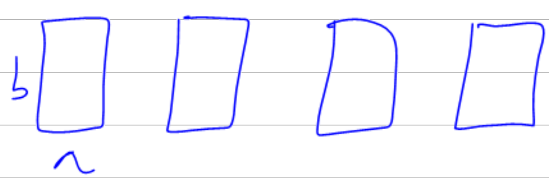
$$|(F(x_1)-F(x_2))-(x_1-x_2)| \le \frac{1}{257}|x_1-x_2|$$

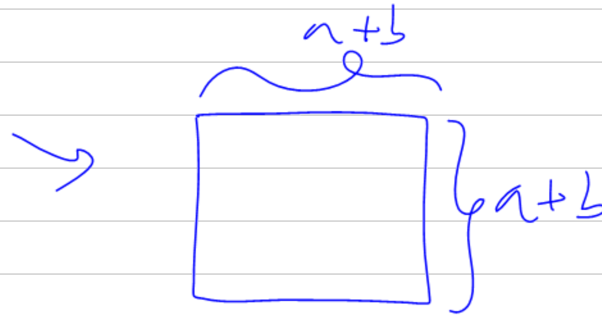



F has "all scale fidelity"

21 ×  in 

4 a×b in $(a+b)^2$



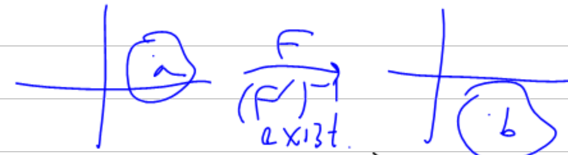




$\Rightarrow$

$$4ab \le (a+b)^2$$

$$\sqrt{ab} \le \frac{a+b}{2}$$ "inequality of the means"

$$\sqrt[3]{abc} \le \frac{a+b+c}{3} \Rightarrow 27abc \le (a+b+c)^3$$ 




**Thm** (The Inverse Function Theorem, IFT)

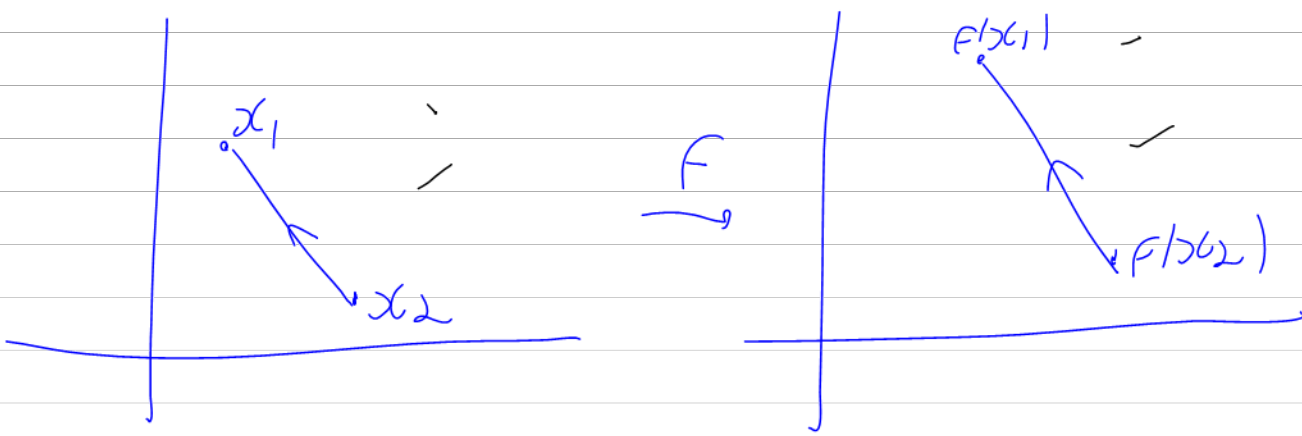
$F:\mathbb{R}^n\to\mathbb{R}^n$ cont. diffable in an open set $A$ containing $a\in\mathbb{R}^n$, $F'(a)$ invertible $\Rightarrow \exists$ open $V\ni a$, open $W\ni b=F(a)$ s.t. $F|_V: V\to W$ is invertible with $F^{-1}:=(F|_V)^{-1}$ is cont., diffable, and with

$$(F^{-1})'(y)=\left[F'(F^{-1}(y))\right]^{-1} \checkmark$$

WLOG, $F'(a)=I$ ✓ Given that, "All Scale Fidelity",

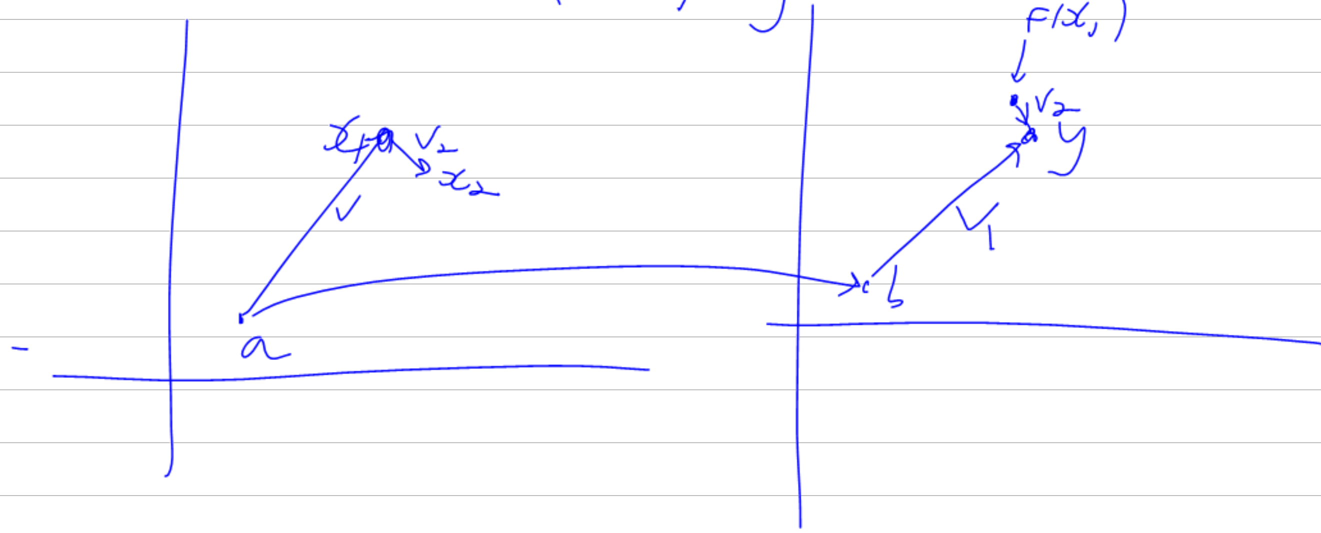
$$\left|(F(x_1)-F(x_2))-(x_1-x_2)\right| < \frac{1}{257}\left|x_1-x_2\right|$$

where $x_1,x_2\in B_r(a)$ $\cancel{\frac{1}{257}}$




Idea

$$F(x)\overset{?}{=}y$$



Let $W = B_{r/2}(b)$ let $y \in W$ $x_0 = a$

$x_1 = a + (y - b) = x_0 + (y - f(x_0))$

$x_2 = x_1 + (y - f(x_1))$

$x_3 = x_2 + (y - f(x_2))$

$\vdots$

$x_{n-1} = x_{n-2} + (y - f(x_{n-2}))$

$x_n = x_{n-1} + (y - f(x_{n-1}))$

$\vdots$

$|x_n - x_{n-1}| =$

$|x_{n-1} + (\cancel{y} - f(x_{n-1})) - x_{n-2} - (\cancel{y} - f(x_{n-2}))|$

$= |(x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2}))|$

by ASF,

$\leq \frac{1}{257} |x_{n-1} - x_{n-2}|$

$\Longrightarrow$ All $x_n \in B_r(a)$ so Fidelity could be used.

Also, $|x_n - x_{n-1}| \leq \left(\frac{1}{257}\right)^n \cdot r$

$\Rightarrow \{x_n\}$ is Cauchy seq:

$\forall \epsilon \; \exists N \; \forall n, m > N \quad |x_n - x_m| < \epsilon$

$\Longrightarrow \exists x$ s.t. $x_n \to x$.

$|f(x_n) - y| = |x_{n+1} - x_n| \leq \frac{r}{257^{n+1}} \longrightarrow 0$

$|f(x) - y| = 0 \Longrightarrow y = f(x)$. ☺

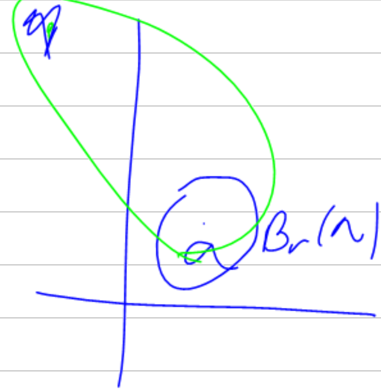
$\hat{W} = B_{r/2}(a)$

Let $V = f^{-1}(W) \cap B_r(a)$ is open

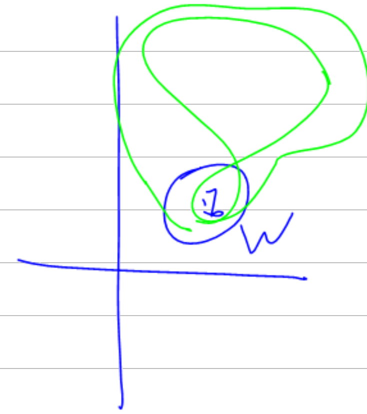
$f|_V : V \longrightarrow W$ onto.

NTS $f|_V$ is 1-1.

Suppose $f(x_1) = f(x_2)$ where $x_1, x_2 \in B_r(a)$









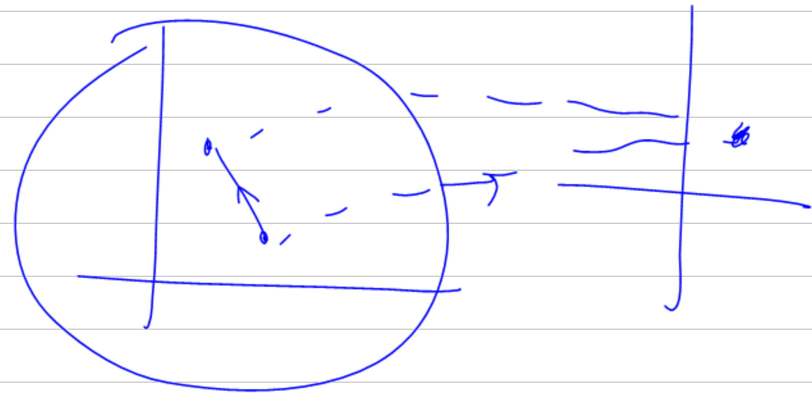
Then use ASF:

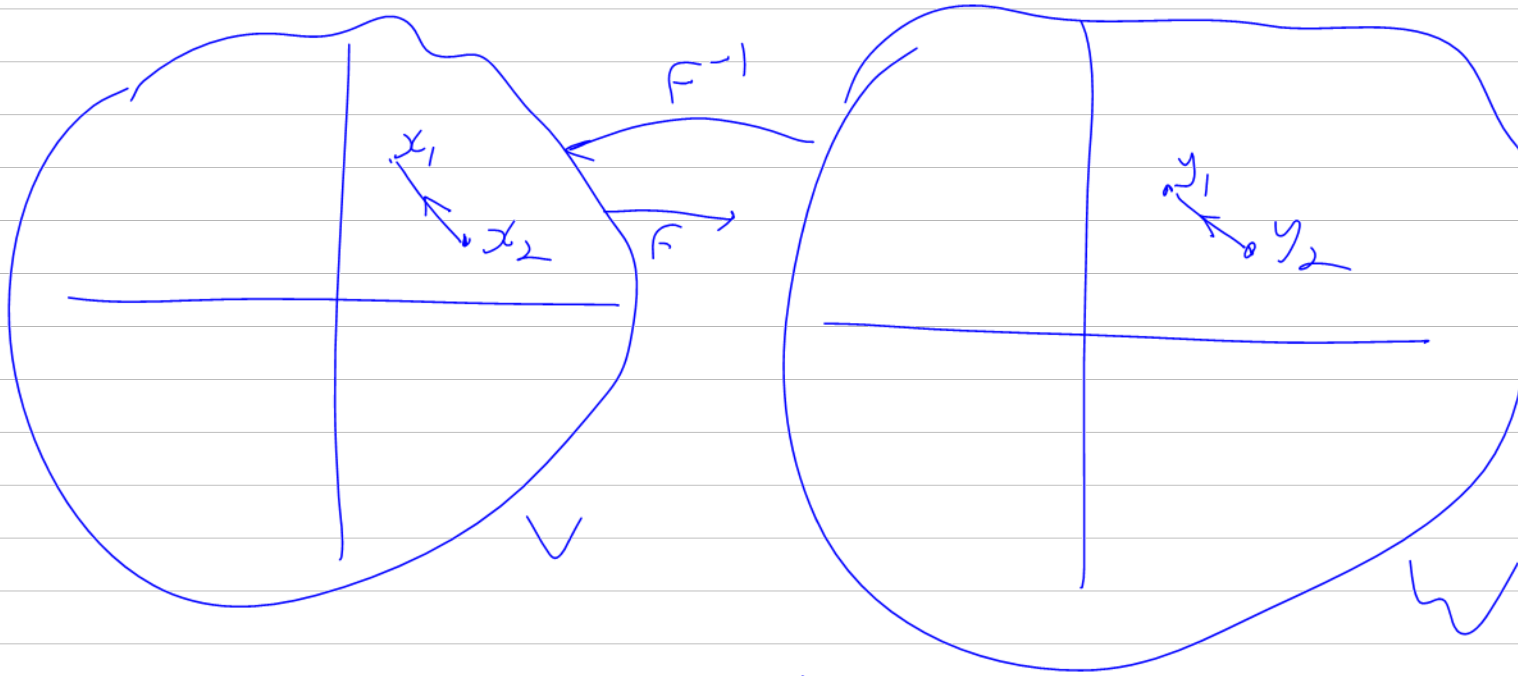
$$|x_1 - x_2 - \cancel{(f(x_1) - f(x_2))}|$$

$$|x_1 - x_2| \leq \frac{1}{257}|x_1 - x_2|$$

$$\Rightarrow |x_1 - x_2| = 0$$

$$\Rightarrow x_1 = x_2 \quad \text{so } f|_V \text{ is 1-1.}$$







Let $y_1, y_2 \in W$, let $x_1 = f^{-1}(y_1) \in V$

$x_2 = f^{-1}(y_2) \in V$

$$|(x_1 - x_2) - (f(x_1) - f(x_2))| \leq \frac{1}{257}|x_1 - x_2|$$

$$|(x_1 - x_2) - (y_1 - y_2)| \leq \frac{1}{257}|x_1 - x_2|$$

Trouble! Fortunately, there's

Monday!