

18-327 on Tuesday November 13, hour 27: The Axiom of Choice,  
Separation Axioms

September-11-10 12:29 PM

Today: Being philosophical around the AC; separation axioms.  
Read Along: Today handout, M31.

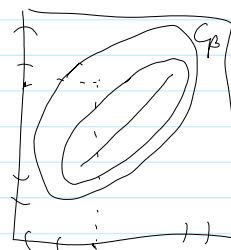
The AC.  $\forall \alpha X_\alpha \neq \emptyset \Rightarrow \prod X_\alpha$  is non-empty;  
alternatively, if  $\pi: X \rightarrow Y$  is surjective, then  
 $\exists \sigma: Y \rightarrow X$  s.t.  $\pi \circ \sigma = I_Y$ .

Zorn's Lemma: In a partially-ordered set in which every chain has a bound, there is a maximal element.

Thm. (Tychonoff) If  $X_\alpha$  is compact for every  $\alpha$ , then so is  $\prod X_\alpha$ .

A quick review of Tychonoff:

Extend a  $\mathcal{C}$  that has the FIP to a maximal  $\mathcal{A}$ .



The Evil Neptune

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## The Evil Neptune

Neptune, the god of the seas, holds an infinite set  $H$  of human prisoners on an island in an ocean far away. One day, he assembles the humans and tells them:

Tomorrow I will place a hat on each of you; black or white it will be, and you will not know which it is. You will be able to see all the hats of all other humans, but if any of you will speak or communicate anything with anybody else, I will know, and you will all be doomed. I will then ask you to all call out at the same time the colour of the hat on your heads, and if all but finitely many of you will get it right, you will be set free. Otherwise [evil laughter]. You have until tomorrow to devise a strategy to counter my threat; I so much enjoy seeing humans squirm [more evil laughter]...

Any suggestions to our unlucky species-mates?

- \* What if all but finitely many hats had the same colour?
- \* What if  $H = \mathbb{Z}$  & Neptune promises to be periodic?

\* What if  $H = \mathbb{Q}$  & Neptune promises to be periodic w/ finitely many exceptions?

\* What if  $H = \mathbb{N}$  so  $\{0,1\}^{\mathbb{N}} = \mathbb{R}$ , & Neptune promises to be rational?

\*  $X = \{0,1\}^H$   $a \sim b \iff |\{h: a(h) \neq b(h)\}| < \infty$  } setup  
 $Y := X/\sim$   $\pi: X \rightarrow Y$

$\sigma: Y \rightarrow X$  s.t.  $\sigma \circ \pi = \pi \circ \sigma = Id_Y$  } strategy

$v \in X$  } Evid.

$v_h(h') = \begin{cases} v(h) & h \neq h' \\ 0 & h = h' \end{cases}$  } survival.  
 $\mu(h) = \sigma(\pi(v_h))(h)$

Indeed  $\pi(v_h) = \pi(v)$  so  $\mu = \sigma(\pi(v_h)) = \sigma(\pi(v))$  so  
 $v \sim \sigma(\pi(v)) = \mu$  !

## A discussion of the A.C. and infinity.

$T_0 \Leftarrow T_1 \Leftarrow T_2 \Leftarrow T_3 \Leftarrow T_4$

$T_3$ /Regular:  $T_1$  & can sep pts & closed sets



$T_4$ /Normal:  $T_1$  & can sep. pairs of closed sets.