18-327 on Thursday September 27, hours 10-11: Closures, T2 Spaces, continuity, products of sets

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Today: closed sets, Hausdorff spaces, a bit more about cont. facts, "the product topology".
Read Along: Munkres 17-19, preread 20-22.
HW1 returned at break/end HW2 due at end HW3 on web late.

$\mathring{A} = \operatorname{int} A =$ biggest contained open

$\bar{A} = \operatorname{cl} A =$ smallest containing closed.

Proposition. $x \in \bar{A}$ iff every (basic) nbd of x intersects A.

Proof. $x \in \bar{A} \iff \begin{matrix}\forall F \text{ closed} \\ F \supset A \Rightarrow x \in F\end{matrix} \iff \begin{matrix}\forall U \text{ open} \\ U \cap A = \emptyset \Rightarrow x \notin U\end{matrix} \iff$

$\iff \begin{matrix}\forall U \text{ open} \\ x \in U \Rightarrow U \cap A \neq \emptyset\end{matrix} \iff [U \text{ (basic) nbd of } x \Rightarrow U \cap A \neq \emptyset]$ $\square$

Definition Limit pt: $x \in A' \iff x \in \overline{A - \{x\}}$, iff every nbd of x contains a point of A other than x itself.

Thm $\bar{A} = A \cup A'$

Pf $\supset$: trivial

$\subset$: $x \in \bar{A} \setminus A \Rightarrow \begin{bmatrix} U \text{ nbd of } x \\ \Rightarrow U \cap A \neq \emptyset \end{bmatrix} \Rightarrow \begin{bmatrix} U \text{ nbd of } x \\ \Rightarrow U \cap A \setminus \{x\} \neq \emptyset \end{bmatrix} \Rightarrow U \in \overline{A - \{x\}}$
$\Rightarrow x \in A'$

Hausdorff spaces. Definition.

Properties. In a T_2 space X:

1. Points are closed.
2. $x \in A'$ iff every nbd contains infinitely many points of x.
3. A sequence converges to at most one limit.
4. Products of T_2 & subspaces of T_2 are T_2.

Continuous Functions. TFAE for $f: X \to Y$:

1. f is cont.
2. $f(\bar{A}) \subset \overline{f(A)}$
3. For every closed $B \subset Y$, $f^{-1}(B)$ is closed in X.
4. For every $x \in X$ & nbd V of $f(x)$, $\exists$ nbd

U of x s.t. $f(U) \subset V$.

Pf $1 \Rightarrow 2$ $x \in \bar{A}$. Take a nbd U of $f(x)$. Then $f^{-1}(U) \cap A \neq \emptyset$

So $U \cap f(A) \neq 0$ so $f(x) \in \overline{f(A)}$.

$2 \Rightarrow 3$ Let $A = f^{-1}(B)$. $f(\bar{A}) \subset \overline{f(f^{-1}(B))} \subset \bar{B} = B$

So $\bar{A} \subset A$ so A is closed.

$3 \Leftrightarrow 1$ ✓

$1 \Leftrightarrow 4$ ✓

The Product Topology. On $\prod_{\alpha \in I} X_\alpha = \{f : I \to \bigcup X_\alpha : f(\alpha) \in X_\alpha\}$

The axiom of choice: "this is not empty".

done line

1. Definition by properties
2. Basis.

The box topology. [on finite products, this is the same]

Example $\mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}}$ by $t \mapsto (t, t \ldots)$ is continuous in cyl but not in box [so box is strictly finer than cyl]

In both box & cyl:

1. The topology on $\prod A_\alpha \subset \prod X_\alpha$ as subspace is the same as ... as a product of subspaces.
2. If X_α is T_2 $\forall \alpha$, then $\prod X_\alpha$ is T_2.
3. $\overline{\prod A_\alpha} = \prod \bar{A}_\alpha$