Solve 4 of the following 5 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1.

1. Define the "finite-complement topology" on a given set X. No need to prove it is a topology.
2. Let X and Y be sets taken with their finite-complement topologies. Prove that a function $f: X \to Y$ is continuous if and only if it is either constant or "finite to one" (meaning that $\forall y \in Y$, $|f^{-1}(y)| < \infty$).

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2.

1. Define "a topological space X is Hausdorff (T_2)".
2. Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$.

Tip. "If and only if" always means that there are two things to prove.

Problem 3. Let B be the set of bounded sequences of real numbers. It is a subset of the set $X = \mathbb{R}^{\mathbb{N}}$ of all sequences of real numbers.

1. Prove that if X is taken with the box topology, then B is both an open and a closed subset.
   Can use $\mathrm{Bd}\, B = \emptyset \Rightarrow B$ is clopen.
2. Prove that if X is taken with the product (cylinders) topology, then B is neither open nor closed.

Problem 4. Given a set X equipped with a metric d, prove that there exists a unique topology on X for which the following two properties hold:

1. For every $x \in X$, the function $f_x: X \to \mathbb{R}$ defined by $f_x(y) = d(x, y)$ is continuous. Cannot use "$d: X \times X \to \mathbb{R}$ is cont.".
2. If Z is any other topological space, and $g: Z \to X$ is a function for which for every $x \in X$ the function $h_x: Z \to \mathbb{R}$ defined by $h_x(z) = d(x, g(z))$ is continuous, then g itself is continuous.

Tip. "There exists a unique" means two things: "there exists", and "if/once exists, it is unique". Both require a proof!

Problem 5. Let (X_n, d_n) be a sequence of metric spaces whose diameters are at most 1: $\forall n \in \mathbb{N}$, $\forall x, y \in X_n$, $d_n(x, y) \leq 1$. Prove that the product $X = \prod_n X_n$ is metrizable. Okay not to prove that $\sup \frac{1}{n} d_n(x_n, y_n)$ is a metric.

Tip. Once you have finished writing an exam, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and maybe even completely rewrite any parts that came out messy.

Good Luck!