Pensieve header: The ${\mathfrak g}_1$ Invariant.

The $\mathfrak{g}_1$ Invariant

Reminder

Make sure that you have Mathematica and that you play with these programs!

The main $\mathfrak{g}_k$ lemma

In $\mathfrak{g}^\epsilon = \langle h, e, l, f\rangle / ([e, l] = -e,\ [f, l] = f,\ [e, f] = h - 2\epsilon l,\ [h, *] = 0)$ and at $\epsilon^{k+1} = 0$, we have

1. $\mathbb{O}\left(e^{\gamma l + \beta e} \mid le\right) = \mathbb{O}\left(e^{\gamma l + e^\gamma \beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(e^{\gamma l + \beta f} \mid f\, l\right) = \mathbb{O}\left(e^{\gamma l + e^\gamma \beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(e^{\beta e + \alpha f + \delta e f} \mid f\, e\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $\nu = (1 + h\,\delta)^{-1}$ and where for any fixed k, $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ is a fixed polynomial of degree at most $4\,k$ in e, $\sqrt{l}$, f, α, β, with scalar coefficients.

Comment. Even better, $\log(\Lambda_k)$ is of degree at most $2\,k + 2$ in said variables.

The Main $\mathfrak{g}_k$ Theorem

Raw Version. The $\mathfrak{g}_k$ invariant of any S-component tangle T can be written in the form $Z(T) = \mathbb{O}\left(\omega\, e^{L+Q+P} \mid \prod_{i\in S} e_i\, l_i\, f_i\right)$, where ω is a scalar (meaning, a rational function in the variables h_i and their exponentials $t_i = e^{h_i}$), where $L = \sum a_{ij}\, h_i\, l_j$ is a balanced quadratic in the variables h_i and l_j with integer coefficients a_{ij}, where $Q = \sum b_{ij}\, e_i\, f_j$ is a balanced quadratic in the variables e_i and f_j with scalar coefficients b_{ij}, and where P is a polynomial in $\{\epsilon,\ e_i,\ l_i,\ f_i\}$ (with scalar coefficients) whose ϵ^d-term is of degree at most $2\,d + 2$ in $\left\{e_i,\ \sqrt{l_i}\ ,\ f_i\right\}$. Furthermore, after setting $h_i = h$ and $t_i = t$ for all i, the invariant $Z(T)$ is poly-time computable.

Partial Proof. Indeed,

0. $R^\pm = ?$, $n^\pm = ?$.
1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l + \beta e} \mid l\, e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \mid e\, l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l + \beta f} \mid f\, l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \mid l\, f\right)$,
3. $\mathbb{O}\left(\mathcal{P}(e, f)\, e^{\beta e + \alpha f + \delta e f} \mid f\, e\right) = \mathbb{O}\left(\nu\, \mathcal{P}(\partial_\beta, \partial_\alpha)\, e^{\nu(-\alpha\beta h + \beta e + \alpha f + \delta e f)} \Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta) \mid e\, f\right)$, with $\nu = (1 + \mathrm{h}\delta)^{-1}$, and $\Lambda_k(\epsilon, e, l, f, \alpha, \beta, \delta)$ as above.

Implementation at $k = 1$

$\mathbb{E}$1n[ω, L, Q, P] stands for $\omega\, e^{L+Q}(1 + \epsilon\, P)$.

```
DP_{x_→D_α_, y_→D_β_}[P_][f_] := Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
```

```
Λ_k_[h_, e_, l_, f_, α_, β_, δ_] := Λ_k[h, e, l, f, α, β, δ] = Module[{λ},
    λ = Normal@Series[e^((f α + e β)/(1 - α β ϵ)) (1 - α β ϵ)^(-2 l + h/ϵ), {ϵ, 0, k}] /. e- → 1;
    Collect[DP_{α→D_f, β→D_e}[λ][e^((f α + e β + e f δ) / (1 + h δ))] /. e- → 1, ϵ, Simplify]];
```

```
ϵ /: ϵ^p_ /; p > 1 := 0;
```

```
CF[𝔼1n[ω_, L_, Q_, P_]] := 𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
𝔼1n /: 𝔼1n[ω1_, L1_, Q1_, P1_] 𝔼1n[ω2_, L2_, Q2_, P2_] := CF@𝔼1n[ω1 ω2, L1 + L2, Q1 + Q2, P1 + P2];
𝔼1n[ω1_, L1_, Q1_, P1_] ≡ 𝔼1n[ω2_, L2_, Q2_, P2_] := Simplify[ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2];
```

0. $R = \mathbb{O}\left(\exp\left(\text{hl} + \frac{e^h - 1}{h}\,\text{ef} + P\right) \mid e \otimes \text{lf}\right)$:

```
𝔼1n[X⁺_{i_,j_}] := 𝔼1n[1, h_i l_j, h_i^-1 (ℯ^h_i - 1) e_i f_j, P⁺];
𝔼1n[X⁻_{i_,j_}] := 𝔼1n[1, -h_i l_j, h_i^-1 (ℯ^-h_i - 1) e_i f_j, P⁻];
𝔼1n[p_Times] := 𝔼1n /@ p;
```

```
𝔼1n[X⁺_{4,1} X⁺_{2,5} X⁺_{6,3}]
```

$\mathbb{E}1\text{n}\Big[1,\ h_4 l_1 + h_6 l_3 + h_2 l_5,\ \frac{1}{h_2 h_4 h_6}\left(-e_6 f_3 h_2 h_4 + e^{h_6} e_6 f_3 h_2 h_4 - e_4 f_1 h_2 h_6 + e^{h_4} e_4 f_1 h_2 h_6 - e_2 f_5 h_4 h_6 + e^{h_2} e_2 f_5 h_4 h_6\right),\ 3 P^+\Big]$

1. $\mathbb{O}\left(\mathcal{P}(l, e)\, e^{\gamma l + \beta e} \mid l e\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta e} \mid e l\right)$,
2. $\mathbb{O}\left(\mathcal{P}(l, f)\, e^{\gamma l + \beta f} \mid f l\right) = \mathbb{O}\left(\mathcal{P}(\partial_\gamma, \partial_\beta)\, e^{\gamma l + e^\gamma \beta f} \mid l f\right)$.

```
NO_{(x:f|e)_i_ l_j_}[𝔼1n[ω_, L_, Q_, P_]] := With[{q = ℯ^γ β x_i + γ l_j},
   CF[𝔼1n[ω, L,
      ℯ^γ β x_i + (Q /. x_i → 0),
      ℯ^-q DP_{l_j→D_γ, x_i→D_β}[P][ℯ^q]
     ] /. {γ → ∂_{l_j} L, β → ∂_{x_i} Q}]];
```

3. $\mathbb{O}\left(e^{\beta \text{e} + \alpha \text{f} + \delta \text{ef}} \mid \text{fe}\right) = \mathbb{O}\left(\nu e^{\nu(-\alpha\beta \text{h} + \beta \text{e} + \alpha \text{f} + \delta \text{ef})} \mid \text{ef}\right)$, with $\nu = (1 + \text{h}\delta)^{-1}$:

```
NO_{f_i_ e_j_→k_}[𝔼1n[ω_, L_, Q_, P_]] := With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
   CF[𝔼1n[ν ω, L,
       q + (Q /. f_i | e_j → 0),
       ℯ^-q DP_{f_i→D_α, e_j→D_β}[P][ℯ^q] + (Λ_1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
     ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
m_{i_,j_→k_}[Z_] := Module[{x, z}, CF[(Z // NO_{f_i e_j→x} // NO_{l_i e_x} // NO_{f_x l_j}) /. z_i|j|x → z_k]]
```

Meta-associativity

```
ζ = 𝔼1n[ω, Sum_{i=1}^{4} Sum_{j=1}^{4} a_{i,j} h_i l_j, Sum_{i=1}^{4} Sum_{j=1}^{4} b_{i,j} e_i f_j, 0]
```

$\mathbb{E}1\text{n}[\omega,\ h_1 l_1 a_{1,1} + h_1 l_2 a_{1,2} + h_1 l_3 a_{1,3} + h_1 l_4 a_{1,4} + h_2 l_1 a_{2,1} + h_2 l_2 a_{2,2} + h_2 l_3 a_{2,3} + h_2 l_4 a_{2,4} + h_3 l_1 a_{3,1} + h_3 l_2 a_{3,2} + h_3 l_3 a_{3,3} + h_3 l_4 a_{3,4} + h_4 l_1 a_{4,1} + h_4 l_2 a_{4,2} + h_4 l_3 a_{4,3} + h_4 l_4 a_{4,4},$
$e_1 f_1 b_{1,1} + e_1 f_2 b_{1,2} + e_1 f_3 b_{1,3} + e_1 f_4 b_{1,4} + e_2 f_1 b_{2,1} + e_2 f_2 b_{2,2} + e_2 f_3 b_{2,3} + e_2 f_4 b_{2,4} + e_3 f_1 b_{3,1} + e_3 f_2 b_{3,2} + e_3 f_3 b_{3,3} + e_3 f_4 b_{3,4} + e_4 f_1 b_{4,1} + e_4 f_2 b_{4,2} + e_4 f_3 b_{4,3} + e_4 f_4 b_{4,4},\ 0]$

```
Short[ξ // m1,2→1, 5] // Timing
```

$$\left\{1.09375,\ \mathbb{E}1n\left[\frac{\omega}{1+h_1 b_{2,1}},\ h_1 l_1 a_{1,1}+h_1 l_1 a_{1,2}+h_1 l_3 a_{1,3}+h_1 l_4 a_{1,4}+h_1 l_1 a_{2,1}+h_1 l_1 a_{2,2}+h_1 l_3 a_{2,3}+h_1 l_4 a_{2,4}+h_3 l_1 a_{3,1}+ h_3 l_1 a_{3,2}+h_3 l_3 a_{3,3}+h_3 l_4 a_{3,4}+h_4 l_1 a_{4,1}+h_4 l_1 a_{4,2}+h_4 l_3 a_{4,3}+h_4 l_4 a_{4,4},\ \frac{e^{\ll 1\gg} e_1 f_1 b_{1,1}+\ll 41\gg+\ll 1\gg}{1+h_1 b_{2,1}},\ \left(4 l_1 b_{2,1}+4 e^{h_1 a_{1,2}+h_1 a_{2,2}+h_3 a_{3,2}+h_4 a_{4,2}} e_1 f_1 b_{1,1} b_{2,1}+4 e^{h_1 a_{\ll 1\gg}+\ll 2\gg+h_4 \ll 1\gg} e_1 f_1 l_1 b_{1,1} b_{2,1}+\ll 270\gg\right)\Big/\left(2\left(1+h_1 b_{2,1}\right)^4\right)\right]\right\}$$

```
Short[lhs = ξ // m1,2→1 // m1,3→1, 5] // Timing
```

$$\left\{136.859,\ \mathbb{E}1n\left[\omega\Big/\left(1+h_1 b_{2,1}+e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1}-h_1^2 b_{2,2} b_{3,1}+h_1 b_{3,2}+h_1^2 b_{2,1} b_{3,2}\right),\ h_1 l_1 a_{1,1}+h_1 l_1 a_{1,2}+h_1 l_1 a_{1,3}+\ll 10\gg+h_4 l_1 a_{4,2}+h_4 l_1 a_{4,3}+h_4 l_4 a_{4,4},\ \frac{\ll 1\gg}{\ll 1\gg},\ \left(4 l_1 b_{2,1}+4 e^{h_1 a_{1,2}+\ll 6\gg+h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1}+4 e^{\ll 1\gg} e_1 f_1 l_1 b_{1,1} b_{2,1}+\ll 11299\gg+\ll 1\gg+2\ll 7\gg b_{4,2}^2- e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2\right)\Big/\left(2\left(1+h_1 b_{2,1}+e^{\ll 1\gg} h_1 b_{3,1}-h_1^2\ll 1\gg b_{\ll 1\gg}+h_1 b_{3,2}+h_1^2 b_{2,1} b_{3,2}\right)^4\right)\right]\right\}$$

```
Short[rhs = ξ // m2,3→2 // m1,2→1, 5] // Timing
```

$$\left\{112.891,\ \mathbb{E}1n\left[\omega\Big/\left(1+h_1 b_{2,1}+e^{h_1 a_{1,2}+h_1 a_{2,2}+h_1 a_{3,2}+h_4 a_{4,2}} h_1 b_{3,1}-h_1^2 b_{2,2} b_{3,1}+h_1 b_{3,2}+h_1^2 b_{2,1} b_{3,2}\right),\ h_1 l_1 a_{1,1}+h_1 l_1 a_{1,2}+h_1 l_1 a_{1,3}+\ll 10\gg+h_4 l_1 a_{4,2}+h_4 l_1 a_{4,3}+h_4 l_4 a_{4,4},\ \frac{\ll 1\gg}{\ll 1\gg},\ \left(4 l_1 b_{2,1}+4 e^{h_1 a_{1,2}+\ll 6\gg+h_4 a_{4,3}} e_1 f_1 b_{1,1} b_{2,1}+4 e^{\ll 1\gg} e_1 f_1 l_1 b_{1,1} b_{2,1}+\ll 11299\gg+\ll 1\gg+2\ll 7\gg b_{4,2}^2- e_4^2 f_4^2 h_1^5 b_{2,2}^2 b_{3,1}^2 b_{3,4}^2 b_{4,2}^2\right)\Big/\left(2\left(1+h_1 b_{2,1}+e^{\ll 1\gg} h_1 b_{3,1}-h_1^2\ll 1\gg b_{\ll 1\gg}+h_1 b_{3,2}+h_1^2 b_{2,1} b_{3,2}\right)^4\right)\right]\right\}$$

```
lhs ≡ rhs
```

True

Profiling

```
<< "C:\\drorbn\\AcademicPensieve\\Projects\\Profile\\Profile.m"
```

This is Profile.m of http://drorbn.net/AcademicPensieve/Projects/Profile/.
  This version: March 2017. Original version: July 1994.

```
CF[𝔼1n[ω_, L_, Q_, P_]] := PP_CF@𝔼1n[Together[ω], Together[L], Together[Q], Together[P]];
NO_(x:f|e)_i_ l_j_[𝔼1n[ω_, L_, Q_, P_]] := PP_NOxl@With[{q = e^γ β x_i + γ l_j},
    CF[𝔼1n[ω, L,
        e^γ β x_i + (Q /. x_i → 0),
        e^-q DP_{l_j→D_γ, x_i→D_β}[P][e^q]
      ] /. {γ → ∂_{l_j} L, β → ∂_{x_i} Q}]];
NO_f_i_ e_j_→k_[𝔼1n[ω_, L_, Q_, P_]] := PP_NOfe@With[{q = ν (-α β h_k + β e_k + α f_k + δ e_k f_k)},
    CF[𝔼1n[ν ω, L,
         q + (Q /. f_i | e_j → 0),
         e^-q DP_{f_i→D_α, e_j→D_β}[P][e^q] + (Λ_1[h_k, e_k, l_k, f_k, α, β, δ] - 1 /. ϵ → 1)
       ] /. ν → (1 + h_k δ)^-1 /. {α → ∂_{f_i} Q /. e_j → 0, β → ∂_{e_j} Q /. f_i → 0, δ → ∂_{f_i,e_j} Q}]];
```

```
BeginProfile[];
Short[ξ // m1,2→1 // m1,3→1]
EndProfile[];
PrintProfile[]
```

$$\mathbb{E}1n\left[\frac{\omega}{1+h_1 b_{2,1}+\ll 1\gg-\ll 1\gg+h_1 b_{\ll 1\gg}+h_1^2 b_{2,1} b_{3,2}},\ h_1 l_1 a_{1,1}+\ll 14\gg+h_4 l_4 a_{4,4},\ \frac{\ll 1\gg}{\ll 1\gg},\ \frac{\ll 1\gg}{2\left(\ll 1\gg\right)^4}\right]$$

```
CF: called 8 times, time in 122.108/122.108
  Parents:
    ( 2)  45.172/ 45.172 under NOfe
    ( 4)  69.874/ 69.874 under NOxl
    ( 2)   7.062/  7.062 under ProfileRoot
NOxl: called 4 times, time in 9.001/78.875
  Parents:
    ( 4)   9.001/ 78.875 under ProfileRoot
  Children:
    ( 4)  69.874/ 69.874 above CF
NOfe: called 2 times, time in 0.173/45.345
  Parents:
    ( 2)   0.173/ 45.345 under ProfileRoot
  Children:
    ( 2)  45.172/ 45.172 above CF
ProfileRoot: called 0 times, time in 0./0.
  Children:
    ( 2)   7.062/  7.062 above CF
    ( 2)   0.173/ 45.345 above NOfe
    ( 4)   9.001/ 78.875 above NOxl
```

Logos Games

```
Λ1[h, e, l, f, α, β, δ]
```

$$1+\frac{1}{2\left(1+h\,\delta\right)^4}\Big(4\,l\left(1+h\,\delta\right)^2\left((\alpha+e\,\delta)\left(\beta+f\,\delta\right)+\delta\left(1+h\,\delta\right)\right)+2\,f\,(\alpha+e\,\delta)\left(1+h\,\delta\right)\left((\alpha+e\,\delta)\left(\beta+f\,\delta\right)+2\,\delta\left(1+h\,\delta\right)\right)+2\,e\left(\beta+f\,\delta\right)\left(1+h\,\delta\right)\left((\alpha+e\,\delta)\left(\beta+f\,\delta\right)+2\,\delta\left(1+h\,\delta\right)\right)-h\left((\alpha+e\,\delta)^2\left(\beta+f\,\delta\right)^2+4\,\delta\,(\alpha+e\,\delta)\left(\beta+f\,\delta\right)\left(1+h\,\delta\right)+2\,\delta^2\left(1+h\,\delta\right)^2\right)\Big)\,\epsilon$$

$$\text{With}\left[\left\{\xi=\frac{h}{t-1}\right\},\ \Lambda_1[h,e,l,f,\alpha,\beta,\delta]\ /.\ \text{Thread}\left[\{h,e,l,f,\alpha,\beta,\delta\}\to\left\{h,\xi\,e,l,f,\alpha,\xi^{-1}\beta,\xi^{-1}\delta\right\}\right]\right]\ //\ \text{Simplify}$$

$$1+\frac{1}{2\,h\left(1+\left(-1+t\right)\delta\right)^4}\left(-1+t\right)\Big(-\left(-1+t\right)(\alpha+e\,\delta)^2\left(\beta+f\,\delta\right)^2-4\left(-1+t\right)\delta\,(\alpha+e\,\delta)\left(\beta+f\,\delta\right)\left(1+\left(-1+t\right)\delta\right)-2\left(-1+t\right)\delta^2\left(1+\left(-1+t\right)\delta\right)^2+4\,l\left(1+\left(-1+t\right)\delta\right)^2\left(\alpha\left(\beta+f\,\delta\right)+\delta\left(1+\left(-1+t\right)\delta+e\left(\beta+f\,\delta\right)\right)\right)+2\,f\,(\alpha+e\,\delta)\left(1+\left(-1+t\right)\delta\right)\left(\alpha\left(\beta+f\,\delta\right)+\delta\left(2+2\left(-1+t\right)\delta+e\left(\beta+f\,\delta\right)\right)\right)+2\,e\left(\beta+f\,\delta\right)\left(1+\left(-1+t\right)\delta\right)\left(\alpha\left(\beta+f\,\delta\right)+\delta\left(2+2\left(-1+t\right)\delta+e\left(\beta+f\,\delta\right)\right)\right)\Big)\,\epsilon$$

Pragmatic Simplifications

$\mathbb{E}1n[\omega, L, Q, P]$ stands for $\omega\, e^{L+Q}(1+\epsilon P)$; $\mathbb{E}1p[\omega, L, Q, P]$ stands for $\omega^{-1}\, e^{L+\omega^{-1} Q}\left(1+\epsilon\,\omega^{-4}\, P\right)$ /. $e_i \to \frac{t_i-1}{h_i} e_i$, all written in $t_i = e^{h_i}$.

$$\mathbb{E}1n[\mathbb{E}1p[\omega\_, L\_, Q\_, P\_]] := CF\left[PowerExpand /@ CF\left[\mathbb{E}1n\left[\omega^{-1}, L, \omega^{-1} Q, \omega^{-4} P\right] /. \; e_{i\_} :\to \frac{t_i - 1}{h_i} e_i /. \; t_{i\_} :\to e^{h_i}\right]\right];$$

$$CF[\mathbb{E}1p[\omega\_, L\_, Q\_, P\_]] := \mathbb{E}1p[Together[\omega], Together[L], Together[Q], Together[P]];$$

$$\mathbb{E}1p[\mathbb{E}1n[\omega\_, L\_, Q\_, P\_]] := CF\left[\mathbb{E}1p\left[\omega^{-1}, L, \omega^{-1} Q, \omega^{-4} P\right] /. \; e_{i\_} :\to \frac{h_i}{t_i - 1} e_i /. \; h_{i\_} :\to Log[t_i]\right];$$

$$\mathbb{E}1p[\omega1\_, L1\_, Q1\_, P1\_] \equiv \mathbb{E}1p[\omega2\_, L2\_, Q2\_, P2\_] := Simplify[\omega1 == \omega2 \wedge L1 == L2 \wedge Q1 == Q2 \wedge P1 == P2];$$

$$X^+_{1,2} \; // \; \mathbb{E}1n$$

$$\mathbb{E}1n\left[1, h_1 l_2, \frac{\left(-1 + e^{h_1}\right) e_1 f_2}{h_1}, P^+\right]$$

$$X^+_{1,2} \; // \; \mathbb{E}1n \; // \; \mathbb{E}1p$$

$$\mathbb{E}1p[1, Log[t_1] \, l_2, e_1 f_2, P^+]$$

$$\zeta = ReplacePart[\zeta, 4 \to P]$$

$$\zeta$$

$$Simplify /@ \mathbb{E}1p[\zeta]$$

$$\mathbb{E}1p[\zeta]$$

$$\zeta \equiv (\mathbb{E}1p[\zeta] \; // \; \mathbb{E}1n)$$

$$\zeta \equiv \mathbb{E}1n[\mathbb{E}1p[\zeta]]$$

$$Simplify /@ \mathbb{E}1n[\mathbb{E}1p @@ \zeta]$$

$$\mathbb{E}1n[\zeta]$$

$$(\mathbb{E}1p @@ \zeta /. \; h_{i\_} :\to Log[t_i]) \equiv (\mathbb{E}1n[\mathbb{E}1p @@ \zeta] \; // \; \mathbb{E}1p)$$

$$\zeta \equiv \mathbb{E}1p[\mathbb{E}1n[\zeta]]$$

$$\mathbb{E}1p[\omega, L, e_1 f_2 + f_1 e_2 + \delta e_1 f_1, 0] \; // \; \mathbb{E}1n$$

$$\mathbb{E}1n\left[\frac{1}{\omega}, L, \frac{1}{\omega h_1 h_2}\left(-e_2 f_1 h_1 + e^{h_2} e_2 f_1 h_1 - \delta e_1 f_1 h_2 + e^{h_1} \delta e_1 f_1 h_2 - e_1 f_2 h_2 + e^{h_1} e_1 f_2 h_2\right), 0\right]$$

$$Simplify /@ \left(\left(\mathbb{E}1p[\omega, L, e_1 f_2 + f_1 e_2 + \delta e_1 f_1, 0] \; // \; \mathbb{E}1n \; // \; NO_{f_1 e_1 \to 0} \; // \; \mathbb{E}1p\right) /. \; \{t_0 \to t_1, h_0 \to h_1, l_0 \to l_1\} /. \; \{e_2 \to \alpha, f_2 \to \beta\}\right)$$

$$\begin{aligned}
\mathbb{E}1p\Big[&-\delta + \omega + \delta t_1, L, e_0 (\beta + \delta f_0) + \frac{\alpha (\beta + \omega f_0 - \beta t_1)}{\omega}, \frac{1}{2 Log[t_1]} \\
&(-1 + t_1) \big(\alpha^2 \beta^2 - 4 \alpha \beta \delta^2 + 2 \delta^4 + 4 \alpha \beta \delta \omega - 4 \delta^3 \omega + 2 \delta^2 \omega^2 + 4 \alpha \beta \delta^2 l_1 - 4 \delta^4 l_1 - 8 \alpha \beta \delta \omega l_1 + 12 \delta^3 \omega l_1 + 4 \alpha \beta \omega^2 l_1 - \\
&\quad 12 \delta^2 \omega^2 l_1 + 4 \delta \omega^3 l_1 - \alpha^2 \beta^2 t_1 + 8 \alpha \beta \delta^2 t_1 - 6 \delta^4 t_1 - 4 \alpha \beta \delta \omega t_1 + 8 \delta^3 \omega t_1 - 2 \delta^2 \omega^2 t_1 - 8 \alpha \beta \delta^2 l_1 t_1 + 12 \delta^4 l_1 t_1 + \\
&\quad 8 \alpha \beta \delta \omega l_1 t_1 - 24 \delta^3 \omega l_1 t_1 + 12 \delta^2 \omega^2 l_1 t_1 - 4 \alpha \beta \delta^2 t_1^2 + 6 \delta^4 t_1^2 - 4 \delta^3 \omega t_1^2 + 4 \alpha \beta \delta^2 l_1 t_1^2 - 12 \delta^4 l_1 t_1^2 + 12 \delta^3 \omega l_1 t_1^2 - \\
&\quad 2 \delta^4 t_1^3 + 4 \delta^4 l_1 t_1^3 + \alpha^2 \delta f_0^2 (-\delta + 2 \omega + \delta t_1) + \delta e_0^2 (\beta + \delta f_0) \left(\beta (-\delta + 2 \omega + \delta t_1) + \delta f_0 (-3 \delta + 4 \omega + 3 \delta t_1)\right) + \\
&\quad 2 \alpha f_0 \left(2 \delta l_1 (-\delta + \omega + \delta t_1)^2 + \omega (\alpha \beta + 2 \delta (-\delta + \omega) + 2 \delta^2 t_1)\right) + \\
&\quad 2 e_0 \Big(\alpha \delta^2 f_0^2 (-2 \delta + 3 \omega + 2 \delta t_1) + 2 \delta f_0 \left(\delta l_1 (-\delta + \omega + \delta t_1)^2 + (-\delta + 2 \omega + \delta t_1) (\alpha \beta + \delta (-\delta + \omega) + \delta^2 t_1)\right) + \\
&\qquad \beta \left(2 \delta l_1 (-\delta + \omega + \delta t_1)^2 + \omega (\alpha \beta + 2 \delta (-\delta + \omega) + 2 \delta^2 t_1)\right)\Big)\big)\Big]
\end{aligned}$$

$$\textbf{Simplify /@ }\left(\left(\mathbb{E}\text{1p}[\omega, L, \omega\, e_1\, \beta + \omega\, f_1\, \alpha + \omega\, \delta\, e_1\, f_1, 0]\ \text{// }\mathbb{E}\text{1n // NO}_{f_1\, e_1 \to 0}\text{ // }\mathbb{E}\text{1p}\right)\ \text{/. }\{t_0 \to t_1, h_0 \to h_1, l_0 \to l_1\}\right)$$

$$\mathbb{E}\text{1p}\Big[\omega\,(1 - \delta + \delta\, t_1),\ L,\ \omega\,\big(e_0\,(\beta + \delta\, f_0) + \alpha\,(\beta + f_0 - \beta\, t_1)\big),$$
$$\frac{1}{2\,\mathrm{Log}[t_1]}\,\omega^4\,(-1 + t_1)\,\big(\alpha^2\beta^2 + 4\alpha\beta\delta + 2\delta^2 - 4\alpha\beta\delta^2 - 4\delta^3 + 2\delta^4 + 4\alpha\beta\, l_1 + 4\delta\, l_1 -$$
$$8\alpha\beta\delta\, l_1 - 12\delta^2 l_1 + 4\alpha\beta\delta^2 l_1 + 12\delta^3 l_1 - 4\delta^4 l_1 - \alpha^2\beta^2 t_1 - 4\alpha\beta\delta\, t_1 - 2\delta^2 t_1 + 8\alpha\beta\delta^2 t_1 +$$
$$8\delta^3 t_1 - 6\delta^4 t_1 + 8\alpha\beta\delta\, l_1 t_1 + 12\delta^2 l_1 t_1 - 8\alpha\beta\delta^2 l_1 t_1 - 24\delta^3 l_1 t_1 + 12\delta^4 l_1 t_1 - 4\alpha\beta\delta^2 t_1^2 -$$
$$4\delta^3 t_1^2 + 6\delta^4 t_1^2 + 4\alpha\beta\delta^2 l_1 t_1^2 + 12\delta^3 l_1 t_1^2 - 12\delta^4 l_1 t_1^2 - 2\delta^4 t_1^3 + 4\delta^4 l_1 t_1^3 + \alpha^2\delta f_0^2\,(2 - \delta + \delta\, t_1) +$$
$$2\alpha f_0\,\big(\alpha\beta + 2\delta - 2\delta^2 + 2\delta^2 t_1 + 2\delta\, l_1\,(1 - \delta + \delta\, t_1)^2\big) + \delta\, e_0^2\,(\beta + \delta f_0)\,\big(\beta\,(2 - \delta + \delta\, t_1) + \delta f_0\,(4 - 3\delta + 3\delta\, t_1)\big) +$$
$$2 e_0\,\Big(\alpha\delta^2 f_0^2\,(3 - 2\delta + 2\delta\, t_1) + \beta\,\big(\alpha\beta + 2\delta - 2\delta^2 + 2\delta^2 t_1 + 2\delta\, l_1\,(1 - \delta + \delta\, t_1)^2\big) +$$
$$2\delta f_0\,\big(\delta\, l_1\,(1 - \delta + \delta\, t_1)^2 + (2 - \delta + \delta\, t_1)\,(\alpha\beta + \delta - \delta^2 + \delta^2 t_1)\big)\Big)\big)\Big]$$

$$\textbf{Simplify /@}$$
$$\left(\left(\mathbb{E}\text{1p}\left[\omega, L, e_1\,\beta + f_1\,\alpha + \delta\, e_1\, f_1 * \left(\frac{(t_1 - 1)}{\mathrm{Log}[t_1]}\right), 0\right]\ \text{// }\mathbb{E}\text{1n // NO}_{f_1\, e_1 \to 0}\text{ // }\mathbb{E}\text{1p}\right)\ \text{/. }\{t_0 \to t_1, h_0 \to h_1, l_0 \to l_1\}\right)$$

$$\mathbb{E}\text{1p}\Big[\frac{\delta + \omega\,\mathrm{Log}[t_1] - 2\delta\, t_1 + \delta\, t_1^2}{\mathrm{Log}[t_1]},\ L,\ e_0\left(\beta + \frac{\delta f_0\,(-1 + t_1)}{\mathrm{Log}[t_1]}\right) + \frac{\alpha\,(\beta + \omega f_0 - \beta\, t_1)}{\omega},$$
$$\frac{1}{2\,\mathrm{Log}[t_1]^5}\,(-1 + t_1)\,\big(2\delta^4 + 4\delta^3\omega\,\mathrm{Log}[t_1] - 4\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 + 2\delta^2\omega^2\,\mathrm{Log}[t_1]^2 - 4\alpha\beta\delta\omega\,\mathrm{Log}[t_1]^3 + \alpha^2\beta^2\,\mathrm{Log}[t_1]^4 -$$
$$4\delta^4 l_1 - 12\delta^3\omega\,\mathrm{Log}[t_1]\, l_1 + 4\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 l_1 - 12\delta^2\omega^2\,\mathrm{Log}[t_1]^2 l_1 + 8\alpha\beta\delta\omega\,\mathrm{Log}[t_1]^3 l_1 - 4\delta\omega^3\,\mathrm{Log}[t_1]^3 l_1 +$$
$$4\alpha\beta\omega^2\,\mathrm{Log}[t_1]^4 l_1 - 14\delta^4 t_1 - 20\delta^3\omega\,\mathrm{Log}[t_1]\, t_1 + 16\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 t_1 - 6\delta^2\omega^2\,\mathrm{Log}[t_1]^2 t_1 +$$
$$8\alpha\beta\delta\omega\,\mathrm{Log}[t_1]^3 t_1 - \alpha^2\beta^2\,\mathrm{Log}[t_1]^4 t_1 + 28\delta^4 l_1 t_1 + 60\delta^3\omega\,\mathrm{Log}[t_1]\, l_1 t_1 - 16\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 l_1 t_1 +$$
$$36\delta^2\omega^2\,\mathrm{Log}[t_1]^2 l_1 t_1 - 16\alpha\beta\delta\omega\,\mathrm{Log}[t_1]^3 l_1 t_1 + 4\delta\omega^3\,\mathrm{Log}[t_1]^3 l_1 t_1 + 42\delta^4 t_1^2 + 40\delta^3\omega\,\mathrm{Log}[t_1]\, t_1^2 -$$
$$24\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 t_1^2 + 6\delta^2\omega^2\,\mathrm{Log}[t_1]^2 t_1^2 - 4\alpha\beta\delta\omega\,\mathrm{Log}[t_1]^3 t_1^2 - 84\delta^4 l_1 t_1^2 - 120\delta^3\omega\,\mathrm{Log}[t_1]\, l_1 t_1^2 +$$
$$24\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 l_1 t_1^2 - 36\delta^2\omega^2\,\mathrm{Log}[t_1]^2 l_1 t_1^2 + 8\alpha\beta\delta\omega\,\mathrm{Log}[t_1]^3 l_1 t_1^2 - 70\delta^4 t_1^3 - 40\delta^3\omega\,\mathrm{Log}[t_1]\, t_1^3 +$$
$$16\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 t_1^3 - 2\delta^2\omega^2\,\mathrm{Log}[t_1]^2 t_1^3 + 140\delta^4 l_1 t_1^3 + 120\delta^3\omega\,\mathrm{Log}[t_1]\, l_1 t_1^3 - 16\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 l_1 t_1^3 +$$
$$12\delta^2\omega^2\,\mathrm{Log}[t_1]^2 l_1 t_1^3 + 70\delta^4 t_1^4 + 20\delta^3\omega\,\mathrm{Log}[t_1]\, t_1^4 - 4\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 t_1^4 - 140\delta^4 l_1 t_1^4 - 60\delta^3\omega\,\mathrm{Log}[t_1]\, l_1 t_1^4 +$$
$$4\alpha\beta\delta^2\,\mathrm{Log}[t_1]^2 l_1 t_1^4 - 42\delta^4 t_1^5 - 4\delta^3\omega\,\mathrm{Log}[t_1]\, t_1^5 + 84\delta^4 l_1 t_1^5 + 12\delta^3\omega\,\mathrm{Log}[t_1]\, l_1 t_1^5 + 14\delta^4 t_1^6 - 28\delta^4 l_1 t_1^6 -$$
$$2\delta^4 t_1^7 + 4\delta^4 l_1 t_1^7 + \alpha^2\delta\,\mathrm{Log}[t_1]^2 f_0^2\,(-1 + t_1)\,(\delta + 2\omega\,\mathrm{Log}[t_1] - 2\delta\, t_1 + \delta\, t_1^2) + \delta\, e_0^2\,\big(\beta\,\mathrm{Log}[t_1] + \delta f_0\,(-1 + t_1)\big)$$
$$(-1 + t_1)\,\big(\beta\,\mathrm{Log}[t_1]\,(\delta + 2\omega\,\mathrm{Log}[t_1] - 2\delta\, t_1 + \delta\, t_1^2) + \delta f_0\,(-1 + t_1)\,(3\delta + 4\omega\,\mathrm{Log}[t_1] - 6\delta\, t_1 + 3\delta\, t_1^2)\big) +$$
$$2\alpha\,\mathrm{Log}[t_1]\, f_0\,\Big(2\delta\, l_1\,(-1 + t_1)\,(\delta + \omega\,\mathrm{Log}[t_1] - 2\delta\, t_1 + \delta\, t_1^2)^2 +$$
$$\omega\,\mathrm{Log}[t_1]\,\big(-2\delta^2 - 2\delta\omega\,\mathrm{Log}[t_1] + \alpha\beta\,\mathrm{Log}[t_1]^2 + 2\delta\,(3\delta + \omega\,\mathrm{Log}[t_1])\, t_1 - 6\delta^2 t_1^2 + 2\delta^2 t_1^3\big)\Big) +$$
$$2 e_0\,\Big(\alpha\delta^2\,\mathrm{Log}[t_1]\, f_0^2\,(-1 + t_1)^2\,(2\delta + 3\omega\,\mathrm{Log}[t_1] - 4\delta\, t_1 + 2\delta\, t_1^2) +$$
$$2\delta f_0\,(-1 + t_1)\,\Big(\delta\, l_1\,(-1 + t_1)\,(\delta + \omega\,\mathrm{Log}[t_1] - 2\delta\, t_1 + \delta\, t_1^2)^2 +$$
$$(\delta + 2\omega\,\mathrm{Log}[t_1] - 2\delta\, t_1 + \delta\, t_1^2)\,\big(-\delta^2 - \delta\omega\,\mathrm{Log}[t_1] + \alpha\beta\,\mathrm{Log}[t_1]^2 + \delta\,(3\delta + \omega\,\mathrm{Log}[t_1])\, t_1 - 3\delta^2 t_1^2 + \delta^2 t_1^3\big)\Big) +$$
$$\beta\,\mathrm{Log}[t_1]\,\Big(2\delta\, l_1\,(-1 + t_1)\,(\delta + \omega\,\mathrm{Log}[t_1] - 2\delta\, t_1 + \delta\, t_1^2)^2 + \omega\,\mathrm{Log}[t_1]$$
$$\big(-2\delta^2 - 2\delta\omega\,\mathrm{Log}[t_1] + \alpha\beta\,\mathrm{Log}[t_1]^2 + 2\delta\,(3\delta + \omega\,\mathrm{Log}[t_1])\, t_1 - 6\delta^2 t_1^2 + 2\delta^2 t_1^3\big)\Big)\Big)\big)\Big]$$