Dror Bar-Natan: Classes: 2017: MAT 1350 AKT:
Handout for January 31, 2017:

Γ-Calculus

Derived from Cheat Sheet Meta-Calculi, in http://drorbn.net/AcademicPensieve/Projects/MetaCalculi/.

σ-calculus. $\sigma_1 * \sigma_2 = \sigma_1 \cup \sigma_2$, $m_c^{ab}(\sigma) = (\sigma \backslash \{a,b\}) \cup (c \to \sigma_a \sigma_b) \,/\, (T_a, T_b \to T_c)$, $\mathrm{tr}_c(\sigma) = \sigma \backslash c$, $R^{\pm}_{ab} \mapsto (a \to 1,\ b \to T_a^{\pm 1})$

Gassner calculus / Γ-calculus. Preserves $C_1 :=$ [col sum = 1] (⇔OC) and ✓ $C_2 :=$ [$\forall a, b$, $(T_a - 1) \mid (A_{ab} - \delta_{ab}\sigma_b)$]

• Except under tr_c, at $T_* = 1$, $\omega = 1$ and $A = I$.

ω	a	b	S
a	α	β	θ
b	γ	δ	ϵ
S	ϕ	ψ	Ξ

$\xrightarrow[\substack{\mu:=1-\beta \\ T_a,T_b\to T_c}]{m_c^{ab}}$

$\mu\omega$	c	S
c	$\gamma + \alpha\delta/\mu$	$\epsilon + \delta\theta/\mu$
S	$\phi + \alpha\psi/\mu$	$\Xi + \psi\theta/\mu$

ω	c	S
c	α	θ
S	ψ	Ξ

$\xrightarrow[\mu:=1-\alpha]{\mathrm{tr}_c}$

$\mu\omega$	S
S	$\Xi + \psi\theta/\mu$

$R^{\pm}_{ab} = \Gamma$

1	a	b
a	1	$1 - T_a^{\pm 1}$
b	0	$T_a^{\pm 1}$

ω	a	S
a	α	θ
S	ϕ	Ξ

$\xrightarrow[\substack{\mu:=T_a-1 \\ \nu:=\alpha-\sigma_a}]{\Delta^a_{bc}}$

ω	b	c	S
b	$(\sigma_a - \alpha T_a - \nu T_c)/\mu$	$(T_b - 1)T_c\nu/\mu$	$(T_b - 1)T_c\theta/\mu$
c	$(T_c - 1)\nu/\mu$	$(\alpha - \sigma_a T_a - \nu T_c)/\mu$	$(T_c - 1)\theta/\mu$
S	ϕ	ϕ	Ξ

$\Big|_{T_a \mapsto T_b T_c}$

Satisfies: ✓ $R^+_{13} /\!\!/ \Delta^1_{12} = R^+_{23} \# R^+_{13}$.
✓ $R^-_{13} /\!\!/ \Delta^1_{12} = R^-_{13} \# R^-_{23}$.
✓ $\Delta^a_{a_1a_2} /\!\!/ \Delta^b_{b_1b_2} /\!\!/ m^{a_1b_1}_{c_1} /\!\!/ m^{a_2b_2}_{c_2} = m^{ab}_c /\!\!/ \Delta^c_{c_1c_2}$.

ω	a	S
a	α	θ
S	ϕ	Ξ

$\xrightarrow{S^a}$

$\alpha\omega/\sigma_a$	a	S
a	$1/\alpha$	θ/α
S	$-\phi/\alpha$	$(\alpha\Xi - \phi\theta)/\alpha$

$\Big|_{T_a \to T_a^{-1}}$

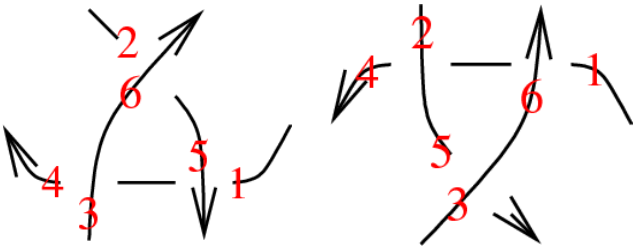
Satisfies: ✓ $R^{\pm}_{12} /\!\!/ S^{1 \text{ or } 2} = R^{\mp}_{12}$.
✓ $S^a /\!\!/ S^a = I$.
✓ Assuming C_2, $\eta^a /\!\!/ \epsilon_a = \Delta^a_{bc} /\!\!/ S^c /\!\!/ m^{bc}_a$ (also 3 variants).
✓ $m^{ab}_c /\!\!/ S^c = S^a /\!\!/ S^b /\!\!/ m^{ba}_c$.
✓ $\Delta^a_{bc} /\!\!/ S^b /\!\!/ S^c = S^a /\!\!/ \Delta^a_{cb}$.

The map (tangle $T \mapsto$ matrix A) is anti-multiplicative.
The MVA mod units: $L \mapsto (\omega, A) \mapsto \omega \det'(A - I)/(1 - T')$ ✓

Pensieve header: Minimal Γ-Calculus, derived from pensieve://2015-07/PolyPoly/nb/Demo.pdf.

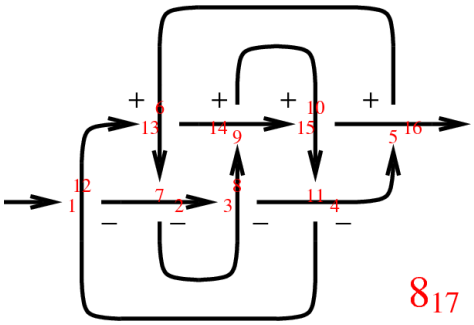
Basics

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_a_ :> a, ∞];
   M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];

Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, Θ, ϵ, ϕ, ψ, Ξ, μ},
   ( α  β  Θ )   ( ∂_{t_a,h_a} λ   ∂_{t_a,h_b} λ   ∂_{t_a} λ )
   ( γ  δ  ϵ ) = ( ∂_{t_b,h_a} λ   ∂_{t_b,h_b} λ   ∂_{t_b} λ ) /. (t | h)_{a|b} → 0;
   ( ϕ  ψ  Ξ )   (   ∂_{h_a} λ       ∂_{h_b} λ        λ     )
   Γ[(μ = 1 - β) ω, {t_c, 1}.( γ + α δ / μ   ϵ + δ Θ / μ ).{h_c, 1}]
                             ( ϕ + α ψ / μ   Ξ + ψ Θ / μ )
     /. {T_a → T_c, T_b → T_c} // ΓCollect];
Rp_{a_b_} := Γ[1, {t_a, t_b}.( 1   1 - T_a ).{h_a, h_b}];
                             ( 0     T_a   )
Rm_{a_b_} := Rp_{ab} /. T_a → 1 / T_a;

ξ = Γ[ω, {t_1, t_2, t_3, t_5}.( α_11  α_12  α_13  Θ_1 ).{h_1, h_2, h_3, h_5}];
                              ( α_21  α_22  α_23  Θ_2 )
                              ( α_31  α_32  α_33  Θ_3 )
                              ( ϕ_1   ϕ_2   ϕ_3   Ξ   )
(ξ // m_{12→1} // m_{13→1}) == (ξ // m_{23→2} // m_{12→1})
True
```



```
{Rm51 Rm62 Rp34 // m14→1 // m25→2 // m36→3,
  Rp61 Rm24 Rm35 // m14→1 // m25→2 // m36→3}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & \frac{T_3}{T_2} & 0 & 0 \\ t_2 & \frac{-1+T_2}{T_2} & \frac{1}{T_3} & 0 \\ t_3 & -\frac{-1+T_3}{T_2} & \frac{-1+T_3}{T_3} & 1 \end{pmatrix}\right\}$$




```
z = Rm12,1 Rm27 Rm83 Rm4,11 Rp16,5 Rp6,13 Rp14,9 Rp10,15;
Do[z = z // m1k→1, {k, 2, 16}]; z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8\,T_1 + 4\,T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$

```
Once[<< KnotTheory`];
Alexander[Knot[8, 17]][T]
```

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

KnotTheory: Loading precomputed data in PD4Knots`.

$$11 - \frac{1}{T^3} + \frac{4}{T^2} - \frac{8}{T} - 8\,T + 4\,T^2 - T^3$$

```
trc_[r[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
   (α θ; ψ Ξ) = (∂tc,hc λ  ∂tc λ; ∂hc λ  λ) /. (t | h)c → 0;
   r[ω (1 - α), Ξ + ψ * θ / (1 - α)] // rCollect];
(ζ // m12→1 // tr1) == (ζ // m21→1 // tr1)
```

```
True
```

Extras

Meta-Associativity in detail

```
Column@{ξ // m12→1, ξ // m23→2, ξ // m12→1 // m13→1, ξ // m23→2 // m12→1}
```

$$\begin{pmatrix} \omega-\omega\,\alpha_{12} & h_1 & h_3 & h_S \\ t_1 & \frac{-\alpha_{21}+\alpha_{12}\,\alpha_{21}-\alpha_{11}\,\alpha_{22}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}}{-1+\alpha_{12}} & \frac{-\alpha_{22}\,\Theta_1-\Theta_2+\alpha_{12}\,\Theta_2}{-1+\alpha_{12}} \\ t_3 & \frac{-\alpha_{31}+\alpha_{12}\,\alpha_{31}-\alpha_{11}\,\alpha_{32}}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\alpha_{32}-\alpha_{33}+\alpha_{12}\,\alpha_{33}}{-1+\alpha_{12}} & \frac{-\alpha_{32}\,\Theta_1-\Theta_3+\alpha_{12}\,\Theta_3}{-1+\alpha_{12}} \\ t_S & \frac{-\phi_1+\alpha_{12}\,\phi_1-\alpha_{11}\,\phi_2}{-1+\alpha_{12}} & \frac{-\alpha_{13}\,\phi_2-\phi_3+\alpha_{12}\,\phi_3}{-1+\alpha_{12}} & \frac{-\Xi+\Xi\,\alpha_{12}-\Theta_1\,\phi_2}{-1+\alpha_{12}} \end{pmatrix}$$

$$\begin{pmatrix} \omega-\omega\,\alpha_{23} & h_1 & h_2 & h_S \\ t_1 & \frac{-\alpha_{11}-\alpha_{13}\,\alpha_{21}+\alpha_{11}\,\alpha_{23}}{-1+\alpha_{23}} & \frac{-\alpha_{12}-\alpha_{13}\,\alpha_{22}+\alpha_{12}\,\alpha_{23}}{-1+\alpha_{23}} & \frac{-\Theta_1+\alpha_{23}\,\Theta_1-\alpha_{13}\,\Theta_2}{-1+\alpha_{23}} \\ t_2 & \frac{-\alpha_{31}+\alpha_{23}\,\alpha_{31}-\alpha_{21}\,\alpha_{33}}{-1+\alpha_{23}} & \frac{-\alpha_{32}+\alpha_{23}\,\alpha_{32}-\alpha_{22}\,\alpha_{33}}{-1+\alpha_{23}} & \frac{-\alpha_{33}\,\Theta_2-\Theta_3+\alpha_{23}\,\Theta_3}{-1+\alpha_{23}} \\ t_S & \frac{-\phi_1+\alpha_{23}\,\phi_1-\alpha_{21}\,\phi_3}{-1+\alpha_{23}} & \frac{-\phi_2+\alpha_{23}\,\phi_2-\alpha_{22}\,\phi_3}{-1+\alpha_{23}} & \frac{-\Xi+\Xi\,\alpha_{23}-\Theta_2\,\phi_3}{-1+\alpha_{23}} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1-\alpha_{13}\,\alpha_{22}+\alpha_{12}\left(-1+\alpha_{23}\right)-\alpha_{23}\right) & h_1 \\ t_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\alpha_{11}\,\alpha_{22}\,\alpha_{33}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ t_S & \frac{\phi_1-\alpha_{12}\,\phi_1-\alpha_{13}\,\alpha_{22}\,\phi_1-\alpha_{23}\,\phi_1+\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{11}\,\phi_2+\alpha_{13}\,\alpha_{21}\,\phi_2-\alpha_{11}\,\alpha_{23}\,\phi_2+\alpha_{21}\,\phi_3-\alpha_{12}\,\alpha_{21}\,\phi_3+\alpha_{11}\,\alpha_{22}\,\phi_3}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1-\alpha_{13}\,\alpha_{22}+\alpha_{12}\left(-1+\alpha_{23}\right)-\alpha_{23}\right) & h_1 \\ t_1 & \frac{\alpha_{31}-\alpha_{12}\,\alpha_{31}-\alpha_{13}\,\alpha_{22}\,\alpha_{31}-\alpha_{23}\,\alpha_{31}+\alpha_{12}\,\alpha_{23}\,\alpha_{31}+\alpha_{11}\,\alpha_{32}+\alpha_{13}\,\alpha_{21}\,\alpha_{32}-\alpha_{11}\,\alpha_{23}\,\alpha_{32}+\alpha_{21}\,\alpha_{33}-\alpha_{12}\,\alpha_{21}\,\alpha_{33}+\alpha_{11}\,\alpha_{22}\,\alpha_{33}}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \\ t_S & \frac{\phi_1-\alpha_{12}\,\phi_1-\alpha_{13}\,\alpha_{22}\,\phi_1-\alpha_{23}\,\phi_1+\alpha_{12}\,\alpha_{23}\,\phi_1+\alpha_{11}\,\phi_2+\alpha_{13}\,\alpha_{21}\,\phi_2-\alpha_{11}\,\alpha_{23}\,\phi_2+\alpha_{21}\,\phi_3-\alpha_{12}\,\alpha_{21}\,\phi_3+\alpha_{11}\,\alpha_{22}\,\phi_3}{1-\alpha_{12}-\alpha_{13}\,\alpha_{22}-\alpha_{23}+\alpha_{12}\,\alpha_{23}} \end{pmatrix}$$

Four types of R1

```
{Rp12 // m12→1, Rp12 // m21→1, Rm12 // m12→1, Rm12 // m21→1}
```

$$\left\{\begin{pmatrix} T_1 & h_1 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{T_1} & h_1 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 \\ t_1 & 1 \end{pmatrix}\right\}$$

Two types of R2

```
{Rp12 Rm34 // m13→1 // m24→2, Rp12 Rm34 // m13→1 // m42→2}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{pmatrix}\right\}$$