Problem 1. Let A be a subset of a metric space (X, d). Show that the distance function to A, defined by $d(x, A) := \inf_{y \in A} d(x, y)$, is a continuous function of x and that $d(x, A) = 0$ iff $x \in \bar{A}$.

Marking key.

Continuity: 13/25.

Suppose $d(x, z) < \epsilon$. As $d(x, A) = \inf_{y \in A} d(x, y)$, there is some $y \in A$ such that $d(x, y) < d(x, A) + \epsilon$, and then $d(z, y) \leq d(z, x) + d(x, y) < \epsilon + d(x, A) + \epsilon = d(x, A) + 2\epsilon$, and so $d(z, A) < d(x, A) + 2\epsilon$. By symmetry also $d(x, A) < d(z, A) + 2\epsilon$, and so $|d(x, A) - d(z, A)| < 2\epsilon$. Hence when $z \to x$ we have that $d(z, A) \to d(x, A)$, so $d(-, A)$ is continuous.

Even better, given x, z, $d(x, A) = \inf_{y \in A} d(x, y) \leq \inf_{y \in A} d(x, z) + dz, y = d(x, z) + \inf_{y \in A} d(z, y) = d(x, z) + d(z, A)$, and by symmetry $d(z, A) \leq d(x, z) + d(x, A)$. So $|d(x, A) - d(z, A) \leq d(x, z)$ and $d(-, A)$ is continuous.

"Iff": 12/25.

Deductions:

(−2) used v.s. notation in a metric space.

(−4) unexplained deduction of the inf inequality from the inequality for a specific point.

(−4) assumed the existence of a minimizer.