

Theorem 2.8. Given a square matrix A, let use reduce it to echelon form B by elementary row operations of types (1) and (2). If B has a zero row, then $\det A = 0$. Otherwise, let k be the number of row exchanges involved in the reduction process. Then $\det A$ equals $(-1)^k$ times the product of the diagonal entries of B.

Definition. Let A be a matrix of size n by m; let B and C be matrices of size m by n. We say that B is a left inverse for A if $B \cdot A = I_m$, and we say that C is a right inverse for A if $A \cdot C = I_n$.

Theorem 2.2. If A has both a left inverse B and a right inverse C, then they are unique and equal.

Theorem 2.7. Let A be a square matrix. If the rows of A are independent, then $\det A \neq 0$; if the rows are dependent, then $\det A = 0$. Thus an n by n matrix A has rank n if and only if $\det A \neq 0$.

Definition. Let A be an n by n matrix. The matrix of size n-1 by n-1 that is obtained from A by deleting the i^{th} row and the j^{th} column of A is called the (i,j)-minor of A. It is denoted A_{ij} . The number

$$(-1)^{i+j} \det A_{ij}$$

Theorem 2.5. If A is a square matrix and if B is a left inverse for A, then B is also a right inverse for A.

Definition. A function that assigns, to each n by n matrix A, a real number denoted det A, is called a **determinant function** if it satisfies the following axioms:

- (1) If B is the matrix obtained by exchanging any two rows of A, then $\det B = -\det A$.
- (2) Given i, the function det A is linear as a function of the ith row alone.
- (3) $\det I_n = 1$.

Theorem 2.4. Let A be a matrix of size n by m. Suppose

$$n = m = rank A$$
.

Then A is invertible; and furthermore, A equals a product of elementary matrices.

Theorem 2.14. Let A be an n by n matrix of rank n; let $B = A^{-1}$. Then

$$b_{ij} = \frac{(-1)^{j+i} \det A_{ji}}{\det A}.$$

Theorem 2.1. Let A be an n by m matrix. Any elementary row operation on A may be carried out by premultiplying A by the corresponding elementary matrix.

Theorem 2.6. Let A be an n by n matrix.

- (a) If E is the elementary matrix corresponding to the operation that exchanges rows i_1 and i_2 , then $det(E \cdot A) = \det A$.
- (b) If E' is the elementary matrix corresponding to the operation that replaces row i_1 of A by itself plus c times row i_2 , then $det(E' \cdot A) = det A$.
- (c) If E'' is the elementary matrix corresponding to the operation that multiplies row i of A by the non-zero scalar λ , then $\det(E'' \cdot A) = \lambda(\det A)$.
 - (d) If A is the identity matrix I_n , then det A = 1.

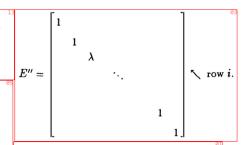
Theorem 2.3. Let A be a matrix of size n by m. If A is invertible, then

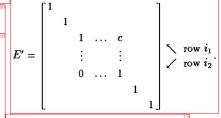
$$n = m = rank A.$$

Definition. If A has both a right inverse and a left inverse, then A is said to be **invertible**. The unique matrix that is both a right inverse and a left inverse for A is called the **inverse** of A, and is denoted A^{-1} .

Theorem 2.15. Let A be an n by n matrix. Let i be fixed. Then

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \cdot \det A_{ik}.$$





Theorem 2.11. det $A^{tr} = \det A$.

$$\det(A \cdot B) = (\det A) \cdot (\det B).$$

Let V be a vector space. A set $\mathbf{a}_1, \dots, \mathbf{a}_m$ of vectors in V is said to span V if to each x in V, there corresponds at least one m-tuple of scalars c_1, \ldots, c_m such that

$$\mathbf{x} = c_1 \mathbf{a}_1 + \cdots + c_m \mathbf{a}_m.$$

In this case, we say that x can be written as a linear combination of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$.

Let V be a vector space of dimension m. If W is Theorem 1.2. a linear subspace of V (different from V), then W has dimension less than m. Furthermore, any basis a_1, \ldots, a_k for W may be extended to a basis $a_1, \ldots, a_k, a_{k+1}, \ldots, a_m$ for V. \square

Theorem 1.6. If B is the matrix obtained by applying an element tary row operation to A, then

$$\operatorname{rank} B = \operatorname{rank} A. \quad \Box$$

If V has a basis consisting of m vectors, we say that m is the dimension. of V. We make the convention that the vector space consisting of the zero vector alone has dimension zero.

The set of matrices has, however, an additional operation, called matrix multiplication. If A is a matrix of size n by m, and if B is a matrix of size m by p, then the product $A \cdot B$ is defined to be the matrix C of size n by p whose general entry c_{ij} is given by the equation

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

The set $\mathbf{a}_1, \dots, \mathbf{a}_m$ of vectors is said to be independent if to each \mathbf{x} is V there corresponds at most one m-tuple of scalars c_1, \ldots, c_m such that

$$\mathbf{x} = c_1 \mathbf{a}_1 + \cdots + c_m \mathbf{a}_m.$$

Equivalently, $\{a_1, \ldots, a_m\}$ is independent if to the zero vector 0 there corresponds only one m-tuple of scalars d_1, \ldots, d_m such that

$$0 = d_1 \mathbf{a}_1 + \cdots + d_m \mathbf{a}_m,$$

namely the scalars $d_1 = d_2 = \cdots = d_m = 0$.

Let V be a vector space with basis $\mathbf{a}_1, \dots, \mathbf{a}_m$. Let Theorem 1.4. W be a vector space. Given any m vectors b_1, \ldots, b_m in W, there is exactly one linear transformation $T: V \to W$ such that, for all i, $T(\mathbf{a}_i) = \mathbf{b}_i$. \square

Suppose V has a basis consisting of m vectors. Theorem 1.1. Then any set of vectors that spans V has at least m vectors, and any set of vectors of V that is independent has at most m vectors. In particular, any basis for V has exactly m vectors. \square

Theorem 1.5. For any matrix A, the row rank of A equals the column rank of A.

(5) For each k, there is a k by k matrix I_k such that if A is any n by mmatrix,

$$I_n \cdot A = A$$
 and $A \cdot I_m = A$

- $I_n \cdot A = A \quad \text{and} \quad A \cdot I_m = A.$ (1) Exchange rows i_1 and i_2 of A (where $i_1 \neq i_2$).
- (2) Replace row i_1 of A by itself plus the scalar c times row i_2 (where $i_1 \neq i_2$).
- (3) Multiply row i of A by the non-zero scalar λ .

Theorem 1.3. If A has size n by m, and B has size m by p, then

$$|A \cdot B| \le m|A||B|$$
. \square

- (1) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
- (2) $A \cdot (B+C) = A \cdot B + A \cdot C$.
- (3) $(A+B)\cdot C = A\cdot C + B\cdot C$.
- $(4) (cA) \cdot B = c(A \cdot B) = A \cdot (cB).$

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n),$$

$$c\mathbf{x} = (cx_1, \dots, cx_n).$$

- (1) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.
- (2) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.
- (3) $\langle c\mathbf{x}, \mathbf{y} \rangle = c \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, c\mathbf{y} \rangle$
- (4) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ if $\mathbf{x} \neq \mathbf{0}$.
- $(1) \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}.$
- $(2) \mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}.$
- (3) There is a unique vector 0
- $(4) \mathbf{x} + (-1)\mathbf{x} = \mathbf{0}.$
- (5) 1x = x.
- (6) $c(d\mathbf{x}) = (cd)\mathbf{x}$.
- $(7) (c+d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}.$
- $(8) c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}.$

$$B = \begin{bmatrix} \textcircled{\textcircled{\textcircled{\otimes}}} & * & * & * & * & * \\ \hline 0 & \textcircled{\textcircled{\textcircled{\textcircled{\textcircled{\o}}}}} & * & * & * & * \\ 0 & 0 & 0 & \textcircled{\textcircled{\textcircled{\textcircled{\o}}}} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- (1) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- (2) $T(c\mathbf{x}) = cT(\mathbf{x})$.

$$C = \begin{bmatrix} \frac{1}{0} & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \dots + x_n y_n^{\perp}$
- (1) $(A^{tr})^{tr} = A$.
- (2) $(A + B)^{tr} = A^{tr} + B^{tr}$
- (3) $(A \cdot C)^{tr} = C^{tr} \cdot A^{tr}$.
- (4) rank $A^{tr} = \text{rank } A$.
- (1) $||\mathbf{x}|| > 0$ if $\mathbf{x} \neq \mathbf{0}$.
- (2) $||c\mathbf{x}|| = |c| ||\mathbf{x}||$.
- $(3) ||\mathbf{x} + \mathbf{y}|| \le ||\mathbf{x}|| + ||\mathbf{y}||.$
- (3') $||\mathbf{x} \mathbf{y}|| \ge ||\mathbf{x}|| ||\mathbf{y}||$

$$|\mathbf{x}| = \max\{|x_1|,\ldots,|x_n|\}^{1/4}$$
 $T(\mathbf{x}) = A \cdot \mathbf{x}^{1/4}$

standard basis for \mathbb{R}^n :0

$$\mathbf{e}_1 = (1,0,0,\ldots,0),$$

$$e_2 = (0, 1, 0, \ldots, 0),$$

$$\mathbf{e}_n = (0,0,0,\ldots,1).$$

$$|\mathbf{x}| \leq ||\mathbf{x}|| \leq \sqrt{n}|\mathbf{x}|^{\frac{1}{2}} ||\mathbf{x}|| = \langle \mathbf{x}, \mathbf{y} \rangle^{1/\frac{1}{2}}$$