1617-257 Wed Oct 5, Hour 11: The Differential

October 5, 2016 7:15 AM

Sec 5 summary: "soft differentiability".

- Directional derivatives f'(a;u).
- Differentiability as "existence of matrix B s.t. some limit vanishes".
- Uniqueness of the differential.
- Directional derivatives in terms of the differential.
- Differentiability implies continuity.
- Differentiability implies that the differential is the matrix of partials.
- If all component functions of f are diffable, so is f.

Read Along: sec 5,6.

on wiki, pagenames/filenames must start w/ 1617-257 !

Today's agenda: ~~differentiability~~ of $f:\mathbb{R}^n\to\mathbb{R}^m$

Riddle along (credit:): Can you write the function

$f(x,y)=1+xy+(xy)^2$ as $f(x,y)=\sum_{k=1}^{2} g_k(x)h_k(y)$?

on board

Reminder: If $f:\mathbb{R}\to\mathbb{R}$ $f'(a):=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

(if the limit exists)

Comment: it is enough that f be defined in a nbd of a. This will be our standing assumption.

Q How do you differentiate $f:\mathbb{R}^n\to\mathbb{R}^m$; e.g.,

$$f_0(x,y)=f_0\binom{x}{y}=\begin{pmatrix}e^x\cos y\\ e^x\sin y\\ (1+x^2+y^2)^{-1}\end{pmatrix}\ ?$$

Naive immitation fails misurably. [First make m>1, then make n>1]

Attempt 1: Directional derivatives, "a 1D view of $\mathbb{R}^n$"

$$f'(a;u):=\lim_{h\to 0}\frac{f(a+hu)-f(a)}{h}$$

Examples

1. $$f_0'\left(0;\binom{1}{1}\right)=\lim_{h\to 0}\frac{f_0(h,h)-f_0(0,0)}{h}=\lim_{h\to 0}h^{-1}\left(\begin{pmatrix}e^h\cos h\\ e^h\sin h\\ (1+2h^2)^{-1}\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}\right)=\begin{pmatrix}(e^h\cos h)'(0)\\ (e^h\sin h)'(0)\\ ((1+2h^2)^{-1})'(0)\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

2. $$f'(a;e_i)=\frac{\partial f}{\partial x_i}(a)=\begin{pmatrix}\partial f_1/\partial x_i(a)\\ \vdots\\ \partial f_m/\partial x_i(a)\end{pmatrix}$$

3. Consider $f(x,y)=\begin{cases}0 & \text{if } (x,y)=0 \text{ or } y\neq x^2\\ 1 & \text{if } y=x^2,\ x\neq 0\end{cases}$

(also, no chain rule)

$f'(a)\sim\frac{f(a+h)-f(a)}{h} \iff f(a+h)\sim f(a)+f'(a)\cdot h$; $\lim_{h\to 0}\frac{|lhs-rhs|}{|h|}=0$

makes sense in every dimension, $f'\to m\times n$ matrix

done

Def $f:\mathbb{R}^n\to\mathbb{R}^m$ is diffable at $a\in\mathbb{R}^n$ if makes sense in every dimension, $f' \to m\times n$ matrix

done line

<u>Def</u> $f: \mathbb{R}^n \to \mathbb{R}^m$ is diffable at $a \in \mathbb{R}^n$ if

$\exists B \in M_{m\times n}$ s.t. $f(a+h) \sim f(a) + Bh$ for small h.

This is the main reason why diffability is important in the natural world!

Two ways to make this precise:

1. The book's way:

$$\frac{|f(a+h) - f(a) - Bh|}{|h|} \xrightarrow[h\to 0]{} 0$$

2. Dror's way: $o(h) := \left\{ g(h) : \lim_{h\to 0} \frac{|g(h)|}{|h|} = 0 \right\}$

$$f(a+h) - f(a) - Bh \in o(h)$$ usually written as

$$f(a+h) = f(a) + Bh + o(h)$$

<u>Theorem:</u> 1. If B exists, it is unique. Call it $Df(a)$, "the differential of f at a".

2. If f is constant, $Df = 0$

3. If $f(x) = Ax$ is linear, $Df(a) = A$.

4. $D(cf) = cDf$ & $D(f+g) = Df + Dg$

5. If f is diffable,

$$f'(a; u) = Df(a) \cdot u$$

and so

$$Df(a) = \begin{pmatrix} \partial f_1/\partial x_1 & & \partial f_1/\partial x_n \\ \vdots & \cdots & \vdots \\ \partial f_m/\partial x_1 & & \partial f_m/\partial x_n \end{pmatrix}$$

"the Jacobbian matrix of f at a".