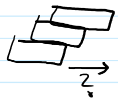
1617-257 Wed Oct 19, Hour 16: Inverse Functions

October 14, 2016 6:25 AM

Riddle Along: How far can you go, with n Jenga blocks? 

Read Along: Sec 7, 8

Agenda: The inverse function theorem!

Chain rule: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$ $\quad D(g\circ f)(a) = (Dg)(f(a))\cdot(Df)(a)$

Claim: If $\gamma \in o(h)$ and $|\delta(x)| < c|x|$ for a fixed c and small x, then $\gamma \circ \delta \in o(x)$

on board.

Pf
$$\frac{|\gamma(\delta(x))|}{|x|} = \frac{|\gamma(\delta(x))|}{|\delta(x)|}\,\frac{|\delta(x)|}{|x|} \longrightarrow 0$$

Often:
$$\mathbb{R}^n_{x_1..x_n} \xrightarrow[\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}]{f} \mathbb{R}^m_{y_1..y_m} \xrightarrow{g} \mathbb{R}$$

$$\frac{\partial}{\partial x_i} g(f_1(x_1 \dots x_n), f_2(x_1 \dots x_n), \dots f_m(x_1 \dots x_n))$$

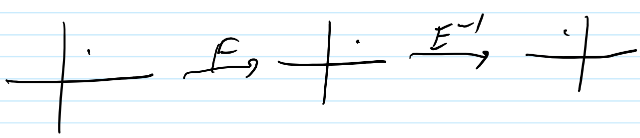
$$= \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_i} + \frac{\partial g}{\partial y_2}\frac{\partial f_2}{\partial x_i} + \dots + \frac{\partial g}{\partial y_m}\frac{\partial f_m}{\partial x_i}$$

Cor 1: If f, g are C^r, so is $g\circ f$.

Corollary 3. If $f, g: \mathbb{R}^n \to \mathbb{R}^n$, f diff'able at a, g diff'able at $b = f(a)$, and $g(f(x)) = x$ near a, then $(Dg)(b) = [Df(a)]^{-1}$

Thm (The Inverse Function Theorem) If $f: \mathbb{R}^n \to \mathbb{R}^n$ is cont. diff'able $\Leftarrow C^1$ near $a \in \mathbb{R}^n$ and $Df(a)$ is invertible, then f is invertible near a; precisely, there are open nbds U of a & V of $b = f(a)$ s.t. $f|_U: U \to V$ is 1-1 & onto. Furthermore, if f is C^r, then so is $f^{-1}: V \to U$.

Comment: WLOG, $Df(a) = I$. Otherwise let $E = Df(a)$, $g = E^{-1}\circ f$ (so $f = E\circ g$)



Then $Dg(a) = I$ & $f^{-1} = g^{-1}\circ E^{-1}$ .....

Technical Lemma (TL): f is "Jelly-rigid" near a: For any x, y near a,

Technical Lemma (TL) f is "Jelly-rigid" near a: For any x, y near a,

$$f(y)-f(x) \sim y-x$$

done fine

precisely, $\forall \epsilon>0\ \exists$ nbd U of a $\forall\ x, y \in U$

$$\|f(y)-f(x)-(y-x)\| \leq \epsilon\|y-x\|$$

False proof

$$f(y)=f(x+(y-x))=f(x)+Df_x(y-x)+\psi(y-x)=f(x)+(I+B)(y-x)+\psi(y-x)$$

where $B\sim\delta$ where $\psi\in o(h)$

So

$$f(y)-f(x)-(y-x) = B(y-x)+\psi(y-x) \quad \left(\text{But } \psi \text{ depends on } x\right)$$

Correct pf MVT to the rescue? $f(b)-f(a)=f'(c)(b-a)$

Aside
MVT in $\mathbb{R}^n$: If $f: \mathbb{R}^n \to \mathbb{R}$ is diffable along the line between a & b, then $\exists c$ on that line s.t.

$$f(b)-f(a)= Df(c)(b-a)$$

Indeed, use 1DMVT on $g(t)=a+t(b-a)$

Back to TL: Find $c_1 \ldots c_n$ between x & y s.t.

$$f_i(y)-f_i(x) = Df(c_i)_i\cdot(y-x) = (I+D_i)_i(y-x) = y_i-x_i+d_i(y-x)$$

where D_i can be made smaller then $\frac{\epsilon}{n}$. Then

$$|f_i(y)-f_i(x)-(y_i-x_i)| = |d_i(y-x)| \leq n\tfrac{\epsilon}{n}|y-x|$$

Part I f is locally 1-1.