1617-257 Wed Nov 9, Hour 24: Continuous functions are integrable, sets of measure 0

October 14, 2016 6:25 AM

Read Along: Secs 11, 1. HW 6 due, HW7 on web by midnight.
Agenda: Continuous functions are integrable, sets of measure 0.
Riddle Along: Are there irrational x,y s.t. x^y is rational?

$m_R(f) = \inf_{x\in R} f(x)$ $L(f,P) = \sum_{R\in P} m_R(f)V(R)$ $\underline{\int_Q} f = \sup_P L(f,P)$

$M_R(f) = \sup_{x\in R} f(x)$ $U(f,P) = \sum_{R\in P} M_R(f)V(R)$ $\overline{\int_Q} f = \inf_P U(f,P)$

Riemann Cond.: f is integrable iff $\forall \epsilon>0\, \exists P$ of Q s.t. $U(f,P)-L(f,P)<\epsilon$.

Thm 2 f unif. cont. $\Rightarrow f$ integrable.

Unif. Cont for $f: X\to Y$: $\forall \epsilon>0\, \exists \delta>0\, \forall x,y\in X$ $d(x,y)<\delta \Rightarrow d(f(x),f(y))<\epsilon$

on board.

Thm 1 Every cont. function on a compact set is uniformly cont.

Prob C of HW2: **Problem C.** Prove the "Lebesgue number lemma": If $\mathcal{U}=\{U_\alpha\}$ is an open cover of a compact space (X,d), then there exists an $\epsilon>0$ (called "the Lebesgue number of $\mathcal{U}$") such that every open ball of radius ϵ in X is contained in one of the U_α's.

Theorem A bdd function $f: Q\to\mathbb{R}$ is integrable iff its set of discontinuities ("disco-set") is of measure 0.

$D = D(f) := \{x\in Q : f \text{ is } \underline{\text{not}} \text{ cont. at } x\}$

Example $\mathbb{1}_A(x) = \begin{cases} 1 & x\in A \\ 0 & x\notin A\end{cases}$ $D(\mathbb{1}_A) = Bd\,A$

Def A set $A \subset\mathbb{R}^n$ is of measure 0 if for every $\epsilon>0$ there is a covering of A with countably many rectangles R_i s.t. $\sum V(R_i)<\epsilon$.

Added Nov 25, 2016: I should have included:
1. The graph of a cont. $f:\mathbb{R}^n\to\mathbb{R}$ is meas-0.
2. A suspension of a meas-0 set is meas-0.

Examples 1. $\mathbb{Q}\subset\mathbb{R}$ is of measure 0.

2. $\{0\}\times[-1,1]^{n-1}\subset[-1,1]^n$ is same

done line

3. For a rectangle Q, if $V(Q)>0$, $Bd(Q)$ is of measure 0 yet Q is not.

Pf that Q is not meas-0: Suppose $\{R_i\}_{i\in I}$ cover Q & $\sum_{i\in I} V(R_i) < V(Q)$.

1. WLOG, $\text{int}(R_i)$ covers $\text{int}(Q)$.
2. WLOG, I is finite
3. WLOG, $\bigcup R_i = Q$

4. Now find a partition P of Q s.t. each R_i is a union of $S_j\in P$, and

$$\sum_{i\in I} V(R_i) = \sum_{i\in I}\sum_{\substack{S\in P\\ S\subset R_i}} V(S) \geq \sum_{S\in P} V(S) = V(Q)$$

Properties: 1. A subset of meas-0 is meas-0.

2. Countable unions

3. coverings by interiors