1617-257 Wed Dec 7, Hour 36: Last class on CofV

December 5, 2016 8:56 AM

Added May 18, 2018: Lax has a very short proof of the change-of-variables formula (see eprints). See also http://drorbn.net/bbs/show?shot=VanDerVeen-180826-141644.jpg.

Save your HW7!!! HW10 due.
Read along: Your notes.
Riddle along: Can you fold a rectangular piece of paper (perhaps many times), so that the result will have a longer perimeter than the original?
Happy holidays and see you next year back at RS 211!

Thm given $A \xrightarrow{g} B \xrightarrow{f} \mathbb{R}$, $\int_B f = \int_A (f\circ g)\cdot|\det Dg|$

Lemma. True for $g_1, g_2 \Rightarrow$ true for $g = g_1 \circ g_2$

Lemma True for affine linear maps: $g(x) = Ax+b$, where $b\in\mathbb{R}^n$ & $A\in M_{n\times n}$ is invertible.

Done for ① $g(x)=x+b$ ② coord. swaps ③ coord. rescales on board

p shears using Fubini. WLOG, $E_c = \begin{pmatrix} 1 & c & \\ & 1 & \\ 0 & & 1\end{pmatrix}$.

Need to show:

$$L = \int_{y\in B} f(y_1\dots y_n) = \int_{x\in A} \underbrace{f(x_1+cx_2, x_2, \dots)}_{f\circ E_c} = R$$

NTS I: f integrable $\Leftrightarrow$ $f\circ E_c$ integrable II. $L=R$.

Start w/ II: write $x=(x_1, x')$. By Fubini,

$$R \overset{\text{Fub}}{=} \int_{x'\in\mathbb{R}^{n-1}}\left(\int_{x_1\in\mathbb{R}} f(x_1+cx_2, x_2, \dots)\right) \overset{(1)}{=} \int_{x'}\int_{x} f(x_1\dots x_n) \overset{\text{Fub}}{=} L.$$

Now I:

$$D(f\circ E_c) = E_c^{-1}(D(f)) = E_{-c}(D(f))$$

Enough, if D is meas-0 in $\mathbb{R}^n$, then so is $E_c(D)$

better, if D is meas-0 in $\mathbb{R}^n$, then so is $L(D)$, where $L:\mathbb{R}^n\to\mathbb{R}^n$ is lin.

enough, given L, $\exists K\in\mathbb{R}$, s.t. for any rectangle R,

$$\text{Vol}(L(R)) \leq K\cdot\text{Vol}(R)$$

Corollary. Let $P(v_1\dots v_n)$ be the parallelepiped in $\mathbb{R}^n$ spanned by $v_1\dots v_n$: done badly

$$P(v_1\dots v_n) = \{\sum a_i v_i : 0\leq a_i\leq 1\}$$ Geometric interp. of dets ?

Then $\text{Vol}(P(v_1\dots v_n)) = |\det(v_1|\dots|v_n)|$

done line

Exercise. Compute the volume of $B^3 = \{x\in\mathbb{R}^3 : |x|\leq 1\}$.

Sol'n use $g(r,\theta,\phi) = (r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)$

$$J = \begin{vmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\phi & 0 & r\cos\phi \end{vmatrix} = \sin^2\phi\cos\phi\cdot r^2 + r\cos^3\phi\cdot r$$

$$= r^2\cos\phi$$

So $V(B^3) = \int_0^1 dr \int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} d\phi\, r^3\cos\phi = \int_0^1 r^2 dr \cdot 4\pi = \frac{4}{3}\pi$