1617-257 Mon Nov 28, Hour 32: Improper Integrals, 2

October 14, 2016 6:25 AM

Read Along: Secs 15. Agenda: More improper integrals
Riddle Along: N prisoners each wears infinitely many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right?

$A \subset \mathbb{R}^k$ open, $f: A \to \mathbb{R}$ cont. $f_\pm = \max(\pm f, 0)$ $f = f_+ - f_-$ $|f| = f_+ + f_-$

$f \geq 0 \Rightarrow \not\int_A f := \sup\{\int_D f : D \subset A \text{ compact and rectifiable}\}$ $\not\int_A f := \not\int_A f_+ - \not\int_A f_-$ if all makes sense

Def: $C_n \nearrow A$ "C_n rises to A" means 1. C_n cmpct & rect. 2. $C_n \subset \operatorname{int} C_{n+1}$ 3. $\bigcup C_n = A$

Thm Given $C_n \nearrow A$, $\not\int_A f$ exist iff $\int_{C_n} |f|$ is bndd, and then $\not\int_A f = \lim_{n\to\infty} \int_{C_n} f$.

on board

Pf of thm. If $f \geq 0$ $\not\int_A f$ implies $\int_{C_n} f \leq \sup_D \int f = \not\int_A f$, so

$\lim_{n\to\infty} \int_{C_n} f$ exists and is $\leq \not\int_A f$.

If $\lim_{n\to\infty} \int_{C_n} f$ exist & $D \subset A$ is compact, then $D \subset C_{n_0}$ for some n_0, hence

$\int_D f \leq \int_{C_{n_0}} f \leq \lim_{n\to\infty} \int_{C_n} f$ so $\not\int_A \leq \lim_{n\to\infty} \int_{C_n} f$ so $\not\int_A f = \lim_{n\to\infty} \int_{C_n} f$.

otherwise $f = f_+ - f_-$ w/ $f_+ \geq 0$, $f_- \geq 0$, and $|f| = f_+ + f_-$ and then

$\not\int_A f$ exists $\Leftrightarrow \not\int_A f_+$ & $\not\int_A f_-$ exist $\Leftrightarrow \lim \int_{C_n} f_+$ & $\lim \int_{C_n} f_-$ exist $\Leftrightarrow \lim \int_{C_n} |f|$ exist

and in that case, $\lim \int_{C_n} f = \lim \int_{C_n} f_+ - f_- = \lim \int_{C_n} f_+ - \lim \int_{C_n} f_- = \not\int_A f_+ - \not\int_A f_- = \not\int_A f$.

Thm If A is open in $\mathbb{R}^n$ and f & g are cont.:

1. $\not\int_A af + bg = a\not\int_A f + b\not\int_A g$ 2. $\int f \leq \int g$ & $|\int f| \leq \int |f|$

3. If $B \subset A$, $\int_B f \leq \int_A f$

4. If A & B are open and f is integrable on A & B,

$$\int_{A\cup B} f = \int_A f + \int_B f - \int_{A\cap B} f$$

Thm If $A \subset \mathbb{R}^n$ is bndd open & $f: A \to \mathbb{R}$ is bndd cont.,

Then $\not\int_A f$ exists. If also $\int_A f$ exists, then $\not\int_A f = \int_A f$.

proof [possibly skip] 1. For $D \subset A$ compact rectifiable, $\int_D |f| \leq \left(\begin{smallmatrix}\text{bound on}\\ f\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix}\text{Vol of}\\ \text{rect containing } A\end{smallmatrix}\right)$,

so $\not\int_A f$ exists.

done [illegible]

so $\not\int_A f$ exists.

done fine

2. If $f \geq 0$, $\int_D f \leq \int_Q f|_A =: \beta\!\!\!\int_A f$, so $\not\int_A f \leq \beta\!\!\!\int_A f$.

Also, for any partition P of Q,

$$L(f|_A, P) = \sum_{R \in P} m_R(f|_A) v(R) = \sum_{\substack{R \in P \\ R \subset A}} m_R(f) v(R) \leq \sum_{\substack{R \in P \\ R \subset A}} \beta\!\!\!\int_R f = \beta\!\!\!\int_D f \leq \not\int_D f$$

where $D = \bigcup_{\substack{R \in P \\ R \subset A}} R$ is compact rectifiable subset of A

so $\beta\!\!\!\int_A f \leq \not\int_A f$, so $\beta\!\!\!\int_A f = \not\int_A f$.

Now if $f = f_+ - f_-$ w/ $f_+ = \max(f, 0)$ & $f_- = \max(-f, 0)$ then f_+ & f_- are integrable so

$$\beta\!\!\!\int_A f = \beta\!\!\!\int_A f_+ - f_- = \beta\!\!\!\int_A f_+ - \beta\!\!\!\int_A f_- = \not\int_A f_+ - \not\int_A f_- = \not\int_A f \qquad \square$$

Corollary. S is bndd & $f: S \to \mathbb{R}$ is bndd cont., then $\beta\!\!\!\int_S f = \not\int_{\text{int} S} f$

<u>Thm.</u> (possibly skip) Let $A \subset \mathbb{R}^n$ be open, $f: A \to \mathbb{R}^n$ cont, $U_1 \subset U_2 \subset \dots$ open sets s.t. $\bigcup U_k = A$. Then $\not\int_A f$ exists iff $\int_{U_k} |f|$ exist and are bndd, and then, $\not\int_A f = \lim_{k \to \infty} \not\int_{U_k} f$.

Examples 1. $f(x,y) = \frac{1}{x^2 y^2}$ on $A = (1, \infty) \times (1, \infty)$

2. Same f on $(0,1) \times (0,1)$.