1617-257 Mon Nov 14, Hour 26: Functions with little discontinuity are integrable

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Read Along: Secs 11, 12.
Agenda: Meas-0 disco-set -> integrable
Riddle Along: Can you find a cont. *onto* f:I->I, which is diffable with Df=0 *except* on a set of measure 0?

Theorem A bndd function $f: Q \to \mathbb{R}$ is integrable iff its disco-set is of measure 0.

$D = D(f) := \{x \in Q : f \text{ is } \underline{\text{not}} \text{ cont. at } x\}$

Def A set $A \subset \mathbb{R}^n$ is of measure 0 if for every $\epsilon > 0$ there is a covering of A with countably many rectangles R_i s.t. $\sum v(R_i) < \epsilon$. on board.

Pf of main thm Assume $|f(x)| \leq M$ on Q.

$\Leftarrow$: Assume $D(f)$ is of meas 0. Let $\epsilon > 0$ [for the Riemann cond.]
Find a countable collection of rectangles Q_i s.t.

$$D(f) \subset \bigcup \operatorname{int} Q_i \quad \& \quad \sum v(Q_i) < \epsilon_1 \quad [\epsilon_1 \text{ TBD}]$$

For each $a \in Q \setminus D(f)$ find a rectangle Q_a s.t. $a \in \operatorname{int} Q_a$ and

$$\sup\{f(x) : x \in Q_a\} - \inf\{f(x) : x \in Q_a\} < \epsilon_2 \quad [\epsilon_2 \text{ TBD}]$$

(Possible because f is cont. at a. Then $\{\operatorname{int} Q_i\} \cup \{\operatorname{int} Q_a\}$ covers Q. Find a finite subcover $\{\operatorname{int} Q_1', \ldots, \operatorname{int} Q_n'\} \subset \{\operatorname{int} Q_i\}$ and $\{\operatorname{int} Q_1'', \ldots, \operatorname{int} Q_m''\} \subset \{\operatorname{int} Q_a\}$, and let P be a partition of Q s.t. each Q_i' and each Q_j'' is a union of rectangles of P. Now

$$U(f,P) - L(f,P) = \sum_{R \in P} v(R)\left(M_R(f) - m_R(f)\right) \leq \overbrace{\sum_{\substack{R \in P \\ R \subset \bigcup Q_i'}} \text{same}}^{A} + \overbrace{\sum_{\substack{R \in P \\ R \subset \bigcup Q_j''}} \text{same}}^{B} = \#$$

$$A \leq \sum_{\substack{R \in P \\ R \subset \bigcup Q_i'}} v(R) \cdot 2M \leq 2M \sum v(Q_i') \leq 2M \sum v(Q_i) < 2M\epsilon_1$$

$$B \leq \sum_{\substack{R \in P \\ R \subset \bigcup Q_j''}} v(R) \cdot \epsilon_2 \leq v(Q)\epsilon_2$$

So $\# < 2M\epsilon_1 + v(Q)\epsilon_2 \leq \epsilon$ if $\epsilon_1 = \epsilon/4M$, $\epsilon_2 = \epsilon/2v(Q)$

$\Rightarrow$ No show; only sketch: 1. Define "oscillation $v(f;a)$"

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2. $D_m = \{a : v(f;a) \geq 1/m\}$ 3. show that D_m is meas-0 $\square$

done line

<u>Theorem</u> Assume $f : Q \to \mathbb{R}$ is integrable.

1. If $f = 0$ almost always (meaning except on a set of meas-0), then $\int_Q f = 0$
2. If $f \geq 0$ and the set $\{x \in Q : f(x) > 0\}$ is not meas-0, then $\int_Q f > 0$.

<u>The Fundamental Thm of Calculus</u> (no proof)

(integration and differentiation are opposites; hyped-up telescopic summation; (hyped-up)^2 "your profit in a week is the sum of your daily profits")

Assume f is cont. on $[a,b]$.

1. If $F(x) := \int_a^x f$, then $F'(x)$ exists & $F'(x) = f(x)$
2. If g is s.t. $g' = f$, the $\int_a^b f = g(b) - g(a)$.

Theorem 12.2 (Fubini's theorem). *Let $Q = A \times B$, where A is a rectangle in $\mathbf{R}^k$ and B is a rectangle in $\mathbf{R}^n$. Let $f : Q \to \mathbf{R}$ be a bounded function; write f in the form $f(\mathbf{x}, \mathbf{y})$ for $\mathbf{x} \in A$ and $\mathbf{y} \in B$. For each $\mathbf{x} \in A$, consider the lower and upper integrals*

$$\underline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \overline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}).$$

If f is integrable over Q, then these two functions of $\mathbf{x}$ are integrable over A, and

$$\int_Q f = \int_{\mathbf{x} \in A} \underline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}) = \int_{\mathbf{x} \in A} \overline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}).$$

Proof. For purposes of this proof, define

$$\underline{I}(\mathbf{x}) = \underline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \overline{I}(\mathbf{x}) = \overline{\int}_{\mathbf{y} \in B} f(\mathbf{x}, \mathbf{y})$$

<u>Exercise</u> 1. Let $Q = [a,b] \times [c,d] \subset \mathbb{R}^2$. Compute $\int_Q e^{2\pi i(x+y)} dx dy$

2. why am I asking?
3. why wasn't it unfair, back then?