1617-257 Fri Oct 7, Hour 12: The Differential, deeper

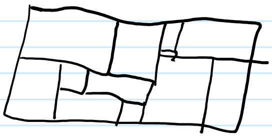
October 5, 2016 7:15 AM

HW1 returned, HW2 due, HW3 on web by midnight.

Today's Agenda: Diffability, deeper.

Read Along: sec 5.1.

Riddle Along: Prove: If you tile a rectangle whose sides are not integers with rectangles, at least one of those will have both sides non-integer:



Reminders: $f'(a,u) := \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$

For $f:\mathbb{R}\to\mathbb{R}$,

$$f(a+h) \sim f(a) + f'(a)\cdot h$$

onboard

Pelucidate:

$$\lim \frac{f(a+h)-f(a)}{h} = f'(a) \Rightarrow \lim \frac{|f(a+h)-f(a)-f'(a)h|}{|h|} = 0$$

Def $f:\mathbb{R}^n \to \mathbb{R}^m$ is diffable at $a\in\mathbb{R}^n$ if

$\exists B \in M_{m\times n}$ s.t. $f(a+h) \sim f(a) + Bh$ for small h.

This is the main reason why Analysis is important in the natural world?

Two ways to make this precise:

1. The book's way:

$$\frac{|f(a+h)-f(a)-Bh|}{|h|} \xrightarrow[h\to 0]{} 0$$

2. Dror's way: $o(h) := \left\{ g(h) : \lim_{h\to 0} \frac{|g(h)|}{|h|} = 0 \right\}$ (should have added $g(0)=0$)

$f(a+h) - f(a) - Bh \in o(h)$ usually written as

$$f(a+h) = f(a) + Bh + o(h)$$

Theorem: 1. If B exists, it is unique. Call it $Df(a)$, "the differential of f at a".

2. If f is constant, $Df = 0$

3. If $f(x) = Ax$ is linear, $Df(a) = A$.

4. $D(cf) = cDf$ & $D(f+g) = Df + Dg$

5. If f is diffable, $f'(a;u) = Df(a)\cdot u$

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and so
$$Df(a) = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_m/\partial x_1 & \cdots & \partial f_m/\partial x_n \end{pmatrix}$$

"the Jacobbian matrix of f at a".

done line

<u>Thm</u> $f:\mathbb{R}^n \to \mathbb{R}$, $\frac{\partial f}{\partial x_i}$ exist and are cont. near a.

Then f is diffable at a. "cont. diffable, class C^1"

Lemma: For any small $h \in \mathbb{R}^n$, $\exists\, q_1 \ldots q_n \in U(a, |h|)$

s.t. $f(a+h) - f(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(q_i)\cdot h_i$

Pf of thm from lemma: With $B = \left(\frac{\partial f}{\partial x_1}(a) \ldots \frac{\partial f}{\partial x_n}(a)\right)$,

$$\frac{f(a+h) - f(a) - B\cdot h}{|h|} = \sum_{i=1}^{n} \frac{\left[\frac{\partial f}{\partial x_i}(q_i) - \frac{\partial f}{\partial x_i}(a)\right] h_i}{|h|}$$

& rhs $\to 0$.

We need a Lemma^2: If $\phi:[a,b]\to\mathbb{R}$ is cont. on $[a,b]$ and diffable in (a,b), then there is a pt. $c\in(a,b)$ s.t.

$$\phi(b) - \phi(a) = \phi'(c)(b-a)$$

This is the mean value thm (MVT) of 157 ☺

Pf of Lemma [given Lemma^2]:

$P_0 = a \quad P_1 = a + h_1 e_1 \quad P_2 = a + h_1 e_1 + h_2 e_2 \quad \ldots \quad P_n = a + \sum_{i=1}^{n} h_i e_i = a+h$

Then
$$f(a+h) - f(a) = \sum_{i=1}^{n} f(P_i) - f(P_{i-1}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(q_i)\cdot h_i$$

if $h_i > 0$, using MVT for
$\phi = f(P_{i-1} + t e_i)$

Is "Prove this Theorem" a fair exam question?