1617-257 Fri Oct 21, Hour 17: Inverse Functions, 2

October 14, 2016 6:25 AM

Riddle Along: Can you place 6 slightly worn Jenga blocks so that any two of them will touch each other?
Read Along: Sec 8.
Agenda: The Inverse Function Theorem.
HW3 returned, HW4 due, HW5 on web by midnight.

<u>Thm</u> (The Inverse Function theorem) If $f:\mathbb{R}^n\to\mathbb{R}^n$ is cont. diffable $\Leftarrow C^1$ near $a\in\mathbb{R}^n$ and $Df(a)$ is invertible, then f is invertible near a; precisely, there are open nbds U of a & V of $b=f(a)$ s.t. $f|_U : U\to V$ is 1-1 & onto. Furthermore, if f is C^r, then so is $f^{-1}: V\to U$.

<u>Comment</u> WLOG, $Df(a)=I$. & $a=b=0$.

<u>Technical Lemma</u> (TL) f is "Jelly-rigid" near a: For any x, y near a,

$$f(y)-f(x)\sim y-x$$

on board

precisely, $\forall\epsilon>0\ \exists$ nbd $J=J_\epsilon=U(0,\delta)$ of 0 s.t.

$$\forall x,y\in J\quad \|f(y)-f(x)-(y-x)\|\le\epsilon\|y-x\|$$

False proof

$$f(y)=f(x+(y-x))=f(x)+Df_x(y-x)+\psi(y-x)=f(x)+(I+B)(y-x)+\psi(y-x)$$

where $B\sim\epsilon$ where $\psi\in o(h)$

So

$$f(y)-f(x)-(y-x)=B(y-x)+\psi(y-x)\qquad \text{(But } \psi \text{ depends on } x\text{)}$$

<u>Correct pf</u> MVT to the rescue? $f(b)-f(a)=f'(c)(b-a)$

<u>Aside</u> MVT in $\mathbb{R}^n$: If $f:\mathbb{R}^n\to\mathbb{R}$ is diffable along the line between a & b, then $\exists c$ on that line s.t.

$$f(b)-f(a)=Df(c)(b-a)$$

Indeed, use 1D MVT on $g(t)=a+t(b-a)$

Back to TL: Find $c_1\dots c_n$ between x & y s.t.

$$f_i(y)-f_i(x)=Df(c_i)_i\cdot(y-x)=(I+D_i)_i(y-x)=y_i-x_i+d_i(y-x)$$

where D_i can be made smaller than $\frac{\epsilon}{n}$. Then

$$|f_i(y)-f_i(x)-(y_i-x_i)|=|d_i(y-x)|\le n\tfrac{\epsilon}{n}|y-x|$$

done

$|F_i(y) - F_i(x) - (y_i - x_i)| = |d_i(y-x)| \leq n\frac{\epsilon}{n}|y-x|$

done line.

<u>Part I</u> f is 1-1 on $J_{0.1}$.

<u>Part II</u> $f|_{J_{0.1}}$ is onto $0.4\,J_{0.1}$. [Let $U = J_{0.1} \cap f^{-1}(0.4 J_{0.1})$ & $V = 0.4 J_{0.1}$]

<u>Part III</u> f^{-1} is cont. on V. (Aside: $|u-v| \leq \epsilon|u| = \epsilon|v+u-v| \leq \epsilon|v| + \epsilon|u-v|$

$(1-\epsilon)|u-v| \leq \epsilon|v|$ so $|u-v| \leq \frac{\epsilon}{1-\epsilon}|v|$)

<u>Part IV</u> f^{-1} is diffable at 0.

<u>Part V</u> f^{-1} is diffable near 0.

<u>Part VI</u> f^{-1} is C^r.