1617-257 Fri Nov 11, Hour 25: sets of measure 0

October 14, 2016 6:25 AM

Read Along: Secs 11, 1.
Agenda: Opinionated propaganda, sets of measure 0.
Riddle Along: Are there irrational x,y s.t. x^y is rational? Discuss.

Theorem A bndd function $f: Q \to \mathbb{R}$ is integrable iff its disco-set is of measure 0.

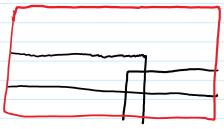
$D = D(f) := \{x \in Q : f \text{ is } \underline{\text{not}} \text{ cont. at } x\}$

Def A set $A \subset \mathbb{R}^n$ is of measure 0 if for every $\epsilon > 0$ there is a covering of A with countably many rectangles R_i s.t. $\sum V(R_i) < \epsilon$. on board.

3. For a rectangle Q, if $V(Q) > 0$, $Bd(Q)$ is of measure 0 yet Q is not.

Pf that Q is not meas 0: Suppose $\{R_i\}_{i \in I}$ cover Q & $\sum_{i \in I} V(R_i) < V(Q)$.

1. WLOG, $\text{int}(R_i)$ covers $\text{int}(Q)$.
2. WLOG, I is finite
3. WLOG, $\bigcup R_i = Q$



4. Now find a partition P of Q s.t. each R_i is a union of $S_i \in P$, and
$$\sum_{i \in I} V(R_i) = \sum_{i \in I} \sum_{\substack{S \in P \\ S \subset R_i}} V(S) \geq \sum_{S \in P} V(S) = V(Q) \qquad \square$$

Aside $\mathbb{Q} \cap I$ cannot be covered by finitely many intervals of total length less than 1.

Properties: 1. A subset of meas-0 is meas-0.
2. Countable unions
3. coverings by interiors

done line

Pf of main thm Assume $|f(x)| \leq M$ on Q.

$\Leftarrow$: Assume $D(f)$ is of meas 0. Let $\epsilon > 0$ [for the Riemann cond.]
Find a countable collection of rectangles Q_i s.t.
$$D(f) \subset \bigcup \text{int}\, Q_i \quad \& \quad \sum V(Q_i) < \epsilon_1 \quad [\epsilon_1 \text{ TBD}]$$
For each $a \in Q \setminus D(f)$ find a rectangle Q_a s.t. $a \in \text{int}\, Q_a$ and
$$\sup\{f(x) : x \in Q_a\} - \inf\{f(x) : x \in Q_a\} < \epsilon_2 \quad [\epsilon_2 \text{ TBD}]$$
(Possible because f is cont. at a. Then $\{\text{int}\, Q_i\} \cup \{\text{int}\, Q_a\}$ covers Q. Find a finite subcover $\{\text{int}\, Q_1', \ldots, \text{int}\, Q_n'\} \subset \{\text{int}\, Q_i\}$

and $\{\operatorname{int} Q_1'' \dots \operatorname{int} Q_m''\} \subset \{\operatorname{int} Q_n\}$, and let $\underline{P}$ be a partition of Q s.t. each Q_i' and each Q_j'' is a union of rectangles of $\underline{P}$. Now

$$U(f,\underline{P}) - L(f,\underline{P}) = \sum_{R\in\underline{P}} v(R)\big(M_R(f)-m_R(f)\big) \leq \overbrace{\sum_{\substack{R\in\underline{P}\\ R\subset\cup Q_i'}} \text{same}}^{A} + \overbrace{\sum_{\substack{R\in\underline{P}\\ R\subset\cup Q_j''}} \text{same}}^{B} = \#$$

$$A \leq \sum_{\substack{R\in\underline{P}\\ R\subset\cup Q_i'}} v(R)\cdot 2M \leq 2M \sum v(Q_i') \leq 2M\sum v(Q_i) < 2M\epsilon_1$$

$$B \leq \sum_{\substack{R\in\underline{P}\\ R\subset\cup Q_j''}} v(R)\cdot\epsilon_2 \leq v(Q)\,\epsilon_2$$

So $\# < 2M\epsilon_1 + v(Q)\epsilon_2 \leq \epsilon$ if $\epsilon_1 = \epsilon/4M$, $\epsilon_2 = \epsilon/2v(Q)$ $\square$