UNIVERSITY OF TORONTO
Faculty of Arts and Sciences
APRIL EXAMINATIONS 2016
Math 475H1 Problem Solving Seminar — Final Exam
April 28, 2016

Solve 8 of the following 11 problems. Indicate clearly which problems you wish graded; otherwise an arbitrary subset of the problems you have attempted will be chosen for marking. The problems carry equal weight.

Duration.   You have 3 hours to write this exam.

Allowed Material.   Stationary only.

This booklet consists of 3 printed pages.

Good Luck!

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 1 (Larson's 1.1.4). Find positive natural numbers n and $a_1, a_2, \ldots, a_n$ such that the sum $a_1 + \cdots + a_n$ is 1000 and the product $a_1 a_2 \cdots a_n$ is as large as possible.
Tip. In math exam, "find" means "find and explain how you found".
Ans: $3^{332} 2^2$ 8/10: Found pattern for A through 10; no proof.
5/10: Optimized the a_i using calculus, for a fixed n.   2/10: Right answer, no justification.
7/10: Correct "lemmas", no proofs.

Problem 2 (Larson's problem 1.2.9). Let $0 < a < b$ be real numbers. If two points are selected at random from a straight line segment of length b, what is the probability that the distance between them is at least a?
Tip. Neatness, cleanliness and organization count, here and everywhere else!
Ans. $(b-a)^2/b^2$.   4/10: Right answer, no justification.
8/10: Correct shading, wrong calculation.   8.5/10: Complementary computation.

Problem 3 (Larson's 1.3.1). Find a general formula for the nth derivative of the function $f(x) = \dfrac{1}{1 - x^2}$.
Ans: $\dfrac{n!}{2}\left(\dfrac{(-1)^n}{(1+x)^{n+1}} + \dfrac{1}{(1-x)^{n+1}}\right)$ 2/10: Brute-computed f', f'' and stopped.

Problem 4 (Larson's 1.4.3). Prove that there does not exist positive integers x, y, and z such that $x^2 + y^2 + z^2 = 2xyz$.
3/10: Correct division into cases, then null.   (-5): unjustified elimination of the (eee) case.
(-2): Eliminated (ooe) before (eee), thus not the general (ooe) case.

Problem 5 (Larson's 1.5.4). Let $-1 < a_0 < 1$ and define recursively for $n > 0$,

$$a_n = \left(\frac{1 + a_{n-1}}{2}\right)^{1/2}.$$

What happens to $4^n(1 - a_n)$ as $n \to \infty$?
Tip. Here and almost always, a concise yet precise solution is better than a lengthy roundabout one.
Sol'n. With $a_0 = \cos\theta$, get $a_n = \cos(\theta/2^n) \sim 1 - \theta^2/2^{2n+1}$. So $\lim = \theta^2/2 = (\arccos a_0)^2/2$.
5/10: Got a_n but not the quadratic cos approximation.

Problem 6. Prove: You cannot colour the points of the plane with just three colours, so that no two points of distance 1 will be coloured with the same colour.
5/10: Constructed a diamond and stopped.

Problem 7 (Larson's 2.6.1). Let n be a natural number. Given $n+1$ positive integers, none of which exceeds $2n$, show that one of these integers divides another of these integers.

Problem 8. A function on the plane has the property that the sum of its values on the corners of any square (of any positive size and orientation) is zero. Prove that the function is the zero function.

Problem 9 (part of Larson's 2.5.13). A *derangement* is a permutation $\sigma \in S_n$ such that for every i, $\sigma i \neq i$. Let g_n be the number of derangements in S_n. Show that

$$g_1 = 0, \quad g_2 = 1, \quad g_n = (n-1)(g_{n-1} + g_{n-2}).$$

Hint. A derangement interchanges 1 with some other element, or not.
Tip. In math-talk, "show" means "prove".

Problem 10 (Larson's 1.12.4c, modified) Compute the sum $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k}$.

Problem 11 (Larson's 1.10.10, reworded). Show that for every positive integer a, the equation $x^2 - y^2 = a^3$ has solutions with $x, y \in \mathbb{Z}$.

Good Luck!