15-344 Combinatorics on Nov 17, hours 26-27: End of Basic Enumeration, Combinatorial Identities

Tuesday, November 17, 2015 9:11 AM

Read Along: 5.5, 6.1.

Agenda. Last enumeration "tutorial", combinatorial identities.

Example 8: Arrangements with Restricted Positions

We return to Example 1 about arrangements of the letters in *banana*, but now with some constraints of the sort encountered in Section 5.2. How many arrangements are there of the letters *b, a, n, a, n, a* such that:

(a) The *b* is followed (immediately) by an *a*:

(b) The pattern *bnn* never occurs:

(c) The *b* occurs before any of the *a*s (not necessarily immediately before an *a*):

Example 1: Assigning Diplomats

How many ways are there to assign 100 different diplomats to five different continents? How many ways if 20 diplomats must be assigned to each continent?

Example 2: Bridge Hands

In bridge, the 52 cards of a standard card deck are randomly dealt 13 apiece to players North, East, South, and West. What is the probability that West has all 13 spades? That each player has one Ace?

Example 3: Distributing Candy

How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?

start line.

1 **Example 4: Distributing a Combination of Identical and Distinct Objects**

How many ways are there to distribute four identical oranges and six distinct apples (each a different variety) into five distinct boxes? In what fraction of these distributions does each box get exactly two objects?

b: Cases: 22000, 21100, 1111 for orange dist.

$$\frac{\binom{5}{2}\cdot\frac{6!}{2!2!2!}+5\cdot\binom{4}{2}\cdot\frac{6!}{2!2!}+5\cdot\frac{6!}{2!}}{\binom{4+4}{4}\cdot 6^5}$$

a: $\binom{4+4}{4}\cdot 6^5$

Example 5: Distributing Balls

Show that the number of ways to distribute r identical balls into n distinct boxes with at least one ball in each box is $C(r-1, n-1)$. With at least r_1 balls in the first box, at least r_2 balls in the second box, ..., and at least r_n balls in the nth box, the number is $C(r-r_1-r_2-\cdots-r_n+n-1, n-1)$.

Example 6: Integer Solutions

2 How many integer solutions are there to the equation $x_1+x_2+x_3+x_4=12$, with $x_i \geq 0$? How many solutions with $x_i \geq 1$? How many solutions with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$?

Example 7: Ingredients for a Witch's Brew

3 A warlock goes to a store with \$5 to buy ingredients for his wife's Witch's Brew. The store sells bat tails for 25¢ apiece, lizard claws for 25¢ apiece, newt eyes for 25¢ apiece, and calf blood for \$1 a pint bottle. How many different purchases (subsets) of ingredients will \$5 buy?

$$\binom{22}{2}+\binom{18}{2}+\binom{14}{2}+\binom{10}{2}+\binom{6}{2}+\binom{2}{2}$$

Example 8: Binary Patterns

What fraction of binary sequences of length 10 consists of a (positive) number of 1s, followed by a number of 0s, followed by a number of 1s, followed by a number of 0s? An example of such a sequence is 1110111000.

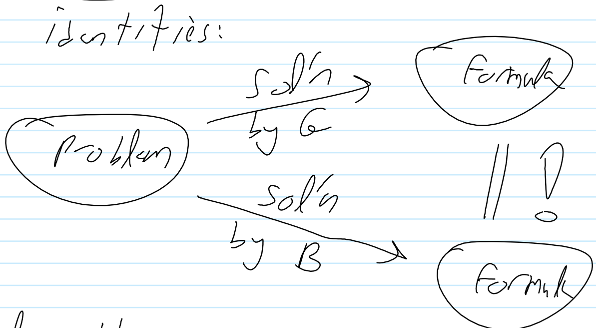
$$\binom{6+3}{3}$$

Example 9: Nonconsecutive Vowels

4 How many arrangements of the letters *a, e, i, o, u, x, x, x, x, x, x, x, x* (eight *x*s) are there if no two vowels can be consecutive?

$$5!\cdot\binom{4+5}{5}$$

Comb. identities:



Example How many ways

for Mik

<u>Example</u> How many ways to choose k of n students?

A: $\binom{n}{k}$ B: $\binom{n}{n-k}$ $=$!

alg. proof...

<u>Aside</u>

$$(a+b)^n = (a+b)(a+b)\cdot\ldots\cdot(a+b) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

e.g. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\uparrow \binom{4}{3}$ $\uparrow \binom{4}{1}$

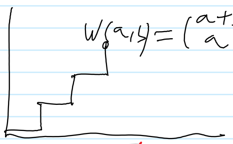
$\binom{n}{k} = \binom{n}{n-k}$ is the "reflection property".

<u>Example</u> How many ways to choose k of n students $\{Amy, \ldots\}$?

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pascal's triangle

Walks on a grid.



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done line.

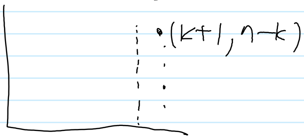
More examples:

1. $\sum \binom{n}{k} = 2^n$ 1. comb. 2. alg.

2. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ 1. comb. 2. alg.

3. $\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}$

1. combinatorial argument. 2. grid argument:



4. $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \ldots + \binom{n+k}{k} = \binom{n+k+1}{k}$

(invent a story)!

5. $\sum \binom{n}{k}^2 = \binom{2n}{n}$

6. $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

The book has 2 more....