15-344 Combinatorics on Nov 12, hour 25: More Enumeration

Wednesday, November 4, 2015 9:02 AM

Read Along 5.1–5.4.

HW7 on web by midnight?

aaaa bbbb cccc dddd: Form a length 10 sequence from this bank, using each letter at least once.

Wrong but tempting: $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\cdot 4^2$

on board.

Count objects = make all objects*, noting all choices.

*making each one just once.

1	2	3	4	5	6	7	8	9	10
c	d	b	a	a	a	c	d	c	b
c	d	b	a	a	a	c	d	c	b

56; 3,10; 1,7; 2,8; a,c

45; 3,10; 1,9; 2,8; a,c

In how many ways did we make each object?

Sol'n w/ $M=\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}=\frac{10!}{2!2!2!2!2!}$

$$M\cdot\frac{4\cdot 3}{3\cdot 3}+M\cdot\frac{4}{6}=2M$$

Alt. (easier)

$$\binom{4}{2}\frac{10!}{3!3!2!2!}+4\frac{10!}{4!2!2!2!}=M\cdot\frac{4\cdot 3}{2}\cdot\frac{2}{3\cdot 3}+M\cdot\frac{4\cdot 2}{3\cdot 4}$$
$$=2M.$$

(not on web)
Dror Bar-Natan: Classes: 2015-16: MAT 344 Combinatorics:

Harder Enumeration

Some Principles from Tuesday November 3.
Sums. If $X \cap Y = \emptyset$ then $|X \cup Y| = |X| + |Y|$.
Products. Always, $|X \times Y| = |X| \cdot |Y|$.
Cancelling Overcounts. If $f\colon X \to Y$ is M-to-1 and onto, then $|X| = M|Y|$.
Words. The number of k-letter words drawn from an alphabet of size n is n^k.
Permutations. The number of permutations of length k (words with no repeating letters) drawn from an alphabet of size n is $P^n_k = n(n-1)(n-2)\dots(n-k+1) = \dfrac{n!}{(n-k)!}$.
In particular, the number of permutations of n letters is $n!$.
Arrangements. The number of arrangements of a_1 objects of type 1, a_2 objects of type 2, …, through a_k objects of type k is $\binom{a_1+a_2+\cdots+a_k}{a_1,\ a_2,\ \dots,\ a_k} = \dfrac{(a_1+a_2+\cdots+a_k)!}{a_1!a_2!\cdots a_k!}$.
Combinations. The number of ways to choose k out of n objects is $C^n_k = \binom{n}{k,\ n-k} = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.
The Candy Bar Principle. The number of ways to divide n identical candies between k kids is $\binom{n+k-1}{k-1}$. "Distributions"?

Example 1: Arrangements of *banana*

How many arrangements are there of the six letters *b, a, n, a, n, a*?

Example 2: Ordering Hot Dogs

How many different ways are there to select six hot dogs from three varieties of hot dog?

Example 3: Grouping Classes

Nine students, three from Ms. A's class, three from Mr. B's class, and three from Ms. C's class, have bought a block of nine seats for their school's homecoming game. If three seats are randomly selected for each class from the nine seats in a row, what is the probability that the three A students, three B students, and three C students will each get a block of three consecutive seats?

Example 4: Sequencing Genes

The genetic code of organisms is stored in DNA molecules as a long string of four nucleotides: A (adenine), C (cytosine), G (guanine), and T (thymine). Short strings of DNA can be "sequenced"—the sequence of letters determined—by various modern biotech methods. Although the DNA sequence for a single gene typically has hundreds or thousands of letters, there exist special enzymes that will split a long string into short fragments (which can be sequenced) by breaking the string immediately following each appearance of a particular letter.

Suppose a C-enzyme (which splits after each appearance of C) breaks a 20-letter string into eight fragments, which are identified to be: AC, AC, AAATC, C, C, C, TATA, TGGC. Note that each fragment, except the last one on the string, must end with a C. How many different strings could have given rise to this set of fragments?

TL;DR?

$\dfrac{7!}{2!3!}$

start line

Example 5: Sequences with Varying Repetitions

1 How many ways are there to form a sequence of 10 letters from four *a*s, four *b*s, four *c*s, and four *d*s if each letter must appear at least twice?

3+3+2+2 : 4+2+2+2 $+ \binom{4}{2}\cdot\dfrac{10!}{3!3!2!2!}$ $4\cdot\dfrac{10!}{4!\,2!2!2!}$

Incorrect: $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}4^2$
Need to show correct soln and explain.

Example 6: Selecting Doughnuts

2 How many ways are there to fill a box with a dozen doughnuts chosen from five different varieties with the requirement that at least one doughnut of each variety is picked?

$\binom{7+4}{4}$

Example 7: Selections with Lower and Upper Bounds

How many ways are there to pick a collection of exactly 10 balls from a pile of red balls, blue balls, and purple balls if there must be at least five red balls? If at most five red balls?

$\binom{5+2}{2}$

$\binom{10+2}{2} - \binom{5+2}{2}$

3

Example 8: Arrangements with Restricted Positions

We return to Example 1 about arrangements of the letters in *banana,* but now with some constraints of the sort encountered in Section 5.2. How many arrangements are there of the letters *b, a, n, a, n, a* such that:

(a) The *b* is followed (immediately) by an *a*:

(b) The pattern *bnn* never occurs:

(c) The *b* occurs before any of the *a*s (not necessarily immediately before an *a*):

a: $\frac{5!}{2!\,2!}$

b: $\frac{6!}{3!\,2!} - \frac{4!}{3!}$

c. $\frac{6!}{3!\,2!\,4} = \frac{6!}{4!\,2!} = 15$

done line

4

Example 1: Assigning Diplomats

How many ways are there to assign 100 different diplomats to five different continents? How many ways if 20 diplomats must be assigned to each continent?

a. 5^{100}

b. $\frac{100!}{(20!)^5}$

5

Example 2: Bridge Hands

In bridge, the 52 cards of a standard card deck are randomly dealt 13 apiece to players North, East, South, and West. What is the probability that West has all 13 spades? That each player has one Ace?

$\frac{39!/(13!)^3}{52!/(13!)^4} = \frac{39!\,13!}{52!}$, $\frac{4!\ 48!/12!^4}{52!/13!^4}$

Example 3: Distributing Candy

How many ways are there to distribute 20 (identical) sticks of red licorice and 15 (identical) sticks of black licorice among five children?

$\binom{24}{4}\cdot\binom{19}{4}$

6

Example 4: Distributing a Combination of Identical and Distinct Objects

How many ways are there to distribute four identical oranges and six distinct apples (each a different variety) into five distinct boxes? In what fraction of these distributions does each box get exactly two objects?

b: Cases: 22200, 21100, 1111 for orange dist.

$\frac{\binom{5}{2}\cdot\frac{6!}{2!\,2!\,2!} + 5\cdot\binom{4}{2}\cdot\frac{6!}{2!\,2!} + 5\cdot\frac{6!}{2!}}{\text{a: } \binom{4+4}{4}\cdot 6^5}$

Example 5: Distributing Balls

Show that the number of ways to distribute r identical balls into n distinct boxes with at least one ball in each box is $C(r-1, n-1)$. With at least r_1 balls in the first box, at least r_2 balls in the second box, . . . , and at least r_n balls in the nth box, the number is $C(r - r_1 - r_2 - \cdots - r_n + n - 1, n - 1)$.

Example 6: Integer Solutions

7 How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$, with $x_i \geq 0$? How many solutions with $x_i \geq 1$? How many solutions with $x_1 \geq 2, x_2 \geq 2, x_3 \geq 4, x_4 \geq 0$?

Example 7: Ingredients for a Witch's Brew

8 A warlock goes to a store with \$5 to buy ingredients for his wife's Witch's Brew. The store sells bat tails for 25¢ apiece, lizard claws for 25¢ apiece, newt eyes for 25¢ apiece, and calf blood for \$1 a pint bottle. How many different purchases (subsets) of ingredients will \$5 buy?

$\binom{22}{2}+\binom{18}{2}+\binom{14}{2}+\binom{10}{2}+\binom{6}{2}+\binom{2}{2}$

Example 8: Binary Patterns

What fraction of binary sequences of length 10 consists of a (positive) number of 1s, followed by a number of 0s, followed by a number of 1s, followed by a number of 0s? An example of such a sequence is 1110111000.

$\binom{6+3}{3}$

Example 9: Nonconsecutive Vowels

9 How many arrangements of the letters $a, e, i, o, u, x, x, x, x, x, x, x, x$ (eight xs) are there if no two vowels can be consecutive?

$5!\cdot\binom{4+5}{5}$