15-344 Combinatorics on Dec 3, hour 34: More Catalan Numbers, some Fibonacci numbers

Monday, November 23, 2015 7:11 AM

Evaluations responses: 31/167 https://course-evals.utoronto.ca/blue/a.aspx?l=591_1709_AAAAAAABgaM

HW 10 on web by midnight!

Agenda: More Catalan, Fibonacci(?) (if time)

Read Along: Your notes, then 7.3, 7.5

Reminders:

$$C_n=\frac{1}{n+1}\binom{2n}{n} \qquad C_n=\sum_{k=1}^{n} C_{n-k}C_{k-1} \implies$$

$$f=f_c=\sum C_k x^k=\frac{1-\sqrt{1-4x}}{2x}$$

$$(1+x)^\alpha=\sum_{k\geq 0}\binom{\alpha}{k}x^k, \quad \binom{\alpha}{k}=\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$$

$$\binom{1/2}{k}=\ldots=\frac{(-1)^{k-1}}{2^k k!}\left(1\cdot 3\cdot 5\cdot\ldots\cdot(2k-3)\right)=\#$$

on board.

Aside: what's $(2n-1)!!$?

Sol'n 1: $(2n-1)!!=\frac{(2n)!}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}=\frac{2n!}{2^n n!}$

Sol'n 2: How many ways to pair $2n$ people?

$$\#=\frac{(-1)^{k-1}}{2^k k!}\,\frac{(2k-2)!}{2^{k-1}(k-1)!}$$

$$=\frac{(-1)^{k-1}}{2^{2k-1}k}\cdot\binom{2k-2}{k-1} \quad \text{w/} \quad \binom{1/2}{0}=1$$

$$\text{So } \sqrt{1-4x}=1+\sum_{k>0}\frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1}\cdot(-4)^k x^k$$

$$=1+\sum_{k>0}\frac{-2}{k}\binom{2k-2}{k-1}x^k=1-2\sum_{k>0}\frac{1}{k}\binom{2k-2}{k-1}x^k$$

So $\frac{1-\sqrt{1-4x}}{2x} = \sum_{k>0} \frac{1}{k}\binom{2k-2}{k-1} x^{k-1}$

$= \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n$!

Fibonacci. How many ways to climb to the top of a staircase on n stairs, climbing one or two stairs each time?

Sol'n $F_0=1$ $F_1=1$ $F_2=2$, $F_n = F_{n-1}+F_{n-2}$

Done line

Generating fnctn:

Method 1: $f = \sum_{k=0}^{\infty} (x+x^2)^k = \frac{1}{1-x-x^2}$

Method 2: $(1-x-x^2)f = 1$

Finding F_n: // instead find G_n, where

$G_0=2$ $G_1=5$ $\qquad G_n = \frac{5}{6}G_{n-1} - \frac{1}{6}G_{n-2}$

$G_2 = 8$ $\qquad (2-x)(3-x) = 6-5x+x^2$

Method 1: Guess $G_n = \alpha^n$

Def A "partition" of n is a way of dividing n identical objects into some number of baskets, whose order is immaterial. P_n is the number of such partitions.

Example: $P_5 = 7$.

5
4+1
3+2
3+1+1

Q what is f_P? I.e., what is $\sum P_n x^n$?

is $\sum_n p_n x^n$?

3 + 2
3 + 1 + 1
2 + 2 + 1
2 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1

Sol'n

$$f_p = (1 + x + x^2 + \dots)(1 + x^2 + x^4 + x^6 + \dots) \cdots$$

$$= \prod_{k=1}^{\infty} \frac{1}{(1 - x^k)}$$