240AlgebraI-141203 Hour 36: Diagonalization Light

December-02-14 10:47 AM

Course evals: 54/112

The Final: All is included, same style as term test & as previous years.

The key: Understand EVERYTHING.

Today: Diagonalization Light.

Tomorrow: Riddles session! Bahen 6183, 10AM.

Problem. For $A \in M_{n\times n}(F)$, compute A^p.

Example. What's $\begin{pmatrix} 4 & 3 \\ -6 & -5 \end{pmatrix}^{15}$?

Observation. If $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, D^p is easy.

Brilliant observation: If $A = CDC^{-1}$, $A^{15} = CD^{15}C^{-1}$

But how do we find C & D?

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad C = \left(v_1 \middle| v_2 \right) \qquad AC = CD \quad (*)$$

Evaluate both sides of $(*)$ on e_1, e_2, get

$$Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_2 v_2$$

$$Av = \lambda v \qquad \text{"eigenvalue eigenvector"}$$

$$\Updownarrow$$

$$(A - \lambda I)v = 0 \iff A - \lambda I \text{ isn't invertible}$$

$$\iff \det(A - \lambda I) = 0$$

$$\iff \det\begin{pmatrix} 4-\lambda & 3 \\ -6 & -5-\lambda \end{pmatrix} = 0$$

$$\iff 0=(4-\lambda)(-5-\lambda)-(-6)3=\lambda^2+\lambda-2$$

$$\iff \lambda=\lambda_{1,2}=1,-2$$

$$\lambda_1=1 \quad \begin{pmatrix} 4-1 & 3 \\ -6 & -5-1 \end{pmatrix}\begin{pmatrix} v_1 \end{pmatrix}=0 \implies v_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2=-2 \quad \begin{pmatrix} 4-(-2) & 3 \\ -6 & -5-(-2) \end{pmatrix}\begin{pmatrix} v_2 \end{pmatrix}=0 \implies v_2=\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\implies C=\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad C^{-1}=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\implies A^5=C\begin{pmatrix} 1 & \\ & -2 \end{pmatrix}^5 C^{-1}=\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -32 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 34 & 33 \\ -66 & -65 \end{pmatrix}$$

Then did http://drorbn.net/AcademicPensieve/Classes/14-240/nb/Fibonacci.pdf

All done!