Office hour today 1200-2200.

\[
\begin{align*}
V & \xrightarrow{T} W \\
Q & \xrightarrow{P} & V' & \xrightarrow{T'} W' \\
E & \xrightarrow{A} & F & \xrightarrow{T_A} G
\end{align*}
\]

\[
\text{rank } T = \text{rank } PTQ \quad \text{rank } [T]^T = \text{rank } T
\]

\[
\text{rank } A = \text{rank } PAQ \text{ where } P \in M_{mn} \text{ & } Q \in M_{nn} \text{ are invertible.}
\]

Look for \( P \times Q \) that will make \( PAQ \) "simpler" than \( A \).

Q1 Which \( P \), \( Q \) ? \( Q2 \) What's simpler?

Ans 2. \( \text{rank } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{rank } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 2 \)

Ans 1. Examples of "good" \( P/Q \): "elementary matrices"

1. Interchanging rows/columns. \( E_{ij} \)
2. Multiplying r/c by a scalar. \( E_{i,j}^{c} \)
3. Adding a multiple of one r/c to another. \( E_{ij}^{\text{r},k} \)

"row/column reactions"

Thm. Every matrix \( A \) can be r/c-reduced to a block matrix of the form \( \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \).
Problem. Find the rank of the matrix

\[ A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \]

Solution. Using (invertible) row/column operations we aim to bring \( A \) to look as close as possible to an identity matrix:

<table>
<thead>
<tr>
<th>Do</th>
<th>Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Bring a 1 to the upper left corner by swapping the first two rows and multiplying the first row (after the swap) by 1/4.</td>
<td>[ \begin{pmatrix} 1 &amp; 1 &amp; 1 &amp; 2 &amp; 0 \ 0 &amp; 2 &amp; 4 &amp; 2 &amp; 2 \ 8 &amp; 2 &amp; 0 &amp; 10 &amp; 2 \ 6 &amp; 3 &amp; 2 &amp; 9 &amp; 1 \end{pmatrix} ]</td>
</tr>
<tr>
<td>2. Add ((-8)) times the first row to the third row, in order to cancel the 3 in position 3-1.</td>
<td>[ \begin{pmatrix} 1 &amp; 1 &amp; 1 &amp; 2 &amp; 0 \ 0 &amp; 2 &amp; 4 &amp; 2 &amp; 2 \ 0 &amp; -6 &amp; -8 &amp; -6 &amp; 2 \ 6 &amp; 3 &amp; 2 &amp; 9 &amp; 1 \end{pmatrix} ]</td>
</tr>
<tr>
<td>3. Likewise add ((-6)) times the first row to the fourth row, in order to cancel the 6 in position 4-1.</td>
<td>[ \begin{pmatrix} 1 &amp; 1 &amp; 1 &amp; 2 &amp; 0 \ 0 &amp; 2 &amp; 4 &amp; 2 &amp; 2 \ 0 &amp; -6 &amp; -8 &amp; -6 &amp; 2 \ 0 &amp; -3 &amp; -4 &amp; -3 &amp; 1 \end{pmatrix} ]</td>
</tr>
<tr>
<td>4. With similar column operations (you need three of those) cancel all the entries in the final row (except, of course, the first, which is used in the canceling).</td>
<td>[ \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 4 &amp; 2 &amp; 2 \ 0 &amp; -6 &amp; -8 &amp; -6 &amp; 2 \ 0 &amp; -3 &amp; -4 &amp; -3 &amp; 1 \end{pmatrix} ]</td>
</tr>
<tr>
<td>5. Turn the 2-2 entry to a 1 by multiplying the second row by 1/2.</td>
<td>[ \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2 &amp; 1 &amp; 1 \ 0 &amp; -6 &amp; -8 &amp; -6 &amp; 2 \ 0 &amp; -3 &amp; -4 &amp; -3 &amp; 1 \end{pmatrix} ]</td>
</tr>
<tr>
<td>6. Using two row operations &quot;clean&quot; the second column; that is, cancel all entries in it other than the &quot;pivot&quot; 1 at position 2-2.</td>
<td>[ \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 2 &amp; 1 &amp; 1 \ 0 &amp; 0 &amp; 4 &amp; 0 &amp; 8 \ 0 &amp; 0 &amp; 2 &amp; 0 &amp; 4 \end{pmatrix} ]</td>
</tr>
<tr>
<td>7. Using three column operations clean the second row except the pivot.</td>
<td>[ \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 4 &amp; 0 &amp; 8 \ 0 &amp; 0 &amp; 2 &amp; 0 &amp; 4 \end{pmatrix} ]</td>
</tr>
<tr>
<td>8. Clean up the row and the column of the 4 in position 3-3 by first multiplying the third row by 1/4 and then performing the appropriate row and column transformations. Notice that by pure luck, the 4 at position 4-5 of the matrix gets killed in action.</td>
<td>[ \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ]</td>
</tr>
</tbody>
</table>

Thus the rank of our matrix is 3.


Claim \( \text{rank } A = \text{rank } (A^T) \) - Btw, the meaning of \( A^T \) in the world of \( A \) is quite intricate.

Claim \( \text{rank } A = \dim (\text{col-space } (A)) = \dim (\text{row-space } (A)) \)

Suppose you could row reduce \( A \) to \( I \). Find \( A^{-1} \).

\[ E_4 E_3 E_2 E_1 A = I \Rightarrow A^{-1} = E_4 E_3 E_2 E_1 \]

* The hard way.

* The easy way: r.r. \( (A | I) \)

Example: Compute \( \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \)

Done (in red).

How far can you go with row reduction? (And then with \( \text{col-space } (A) \))
How far can you go with row reduction?

1. The first non-zero entry in each row ("the pivot") is a 1.
2. In the column of a pivot, all else is 0.
   [Scan from left to right, to prevent interference]
3. Going down the rows, the pivots are further & further to the right.

Example:

\[
\begin{bmatrix}
1 & 0 & 2 & 9 & 0 & 8 \\
0 & 1 & 3 & 7 & 0 & 15 \\
0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

non-zero, non-zero, non-zero, non-zero, non-zero, non-zero rows.

\[
\begin{bmatrix}
1 & 0 & 2 & 9 & 0 & 8 \\
0 & 1 & 3 & 7 & 0 & 15 \\
0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

non-zero, non-zero, non-zero, non-zero, non-zero, non-zero rows.

pivotal cols.
non-pivotal cols.

... And now with col. ops., can reach \(\begin{bmatrix} I_e & 0 \\
0 & 0 \end{bmatrix}\)

Claim: The rank of a r.r.e.f. matrix is the number of pivots/non-zero rows in it.

Claim: If \(A\) is invertible, its r.r.e.f. is \(I\)