Last Wednesday: In the discussion of degrees the two manifolds involved must be "closed", meaning compact and having no boundary.

\[ I = \frac{1}{4\pi} \int \alpha^* W = \frac{1}{4\pi} \int_{S^1} \alpha^* W \]

\[ C_2(S^1) = \{ (x, y) : x + y \in S^1 \} = S^1 \times (0, 1) \]

\[ \text{chose a "swaddling map" } \beta : D^2 \to S^2 \]

\[ \begin{array}{c}
\text{s.t. } \beta|_{S^1} = \gamma \\
\text{and } S^1 \xrightarrow{\beta} S^2 \\
\text{also } \beta \to S^2 \\
\end{array} \]

\[ \text{def } \text{sl}(\gamma, \beta) := \deg \beta = \int_{S^2} \beta^* W \in \mathbb{Z} \]

\[ \text{sl}(\gamma, \beta) = 1 \]

\[ \text{in general, } \text{sl}(\gamma, \beta) \text{ is an odd integer.} \]

\[ \text{As a function of } \gamma \text{ alone, it is defined up on an even integer.} \]

\[ \eta(\gamma) := \frac{1}{4\pi} \int \alpha^* W \]

\[ \text{not compact!} \]

Instead consider \[ \text{sl}_b(\gamma) := l(\gamma, \gamma^B) \]

\[ \gamma^B(s) := \gamma(s) + c \gamma(s) \text{ framing, framed knots,} \]

\[ \text{framings } \sim \mathbb{Z} \text{ as an affine set!} \]

\[ \text{compare two framings by a map } \gamma : S^1 \to S^2 \]
$s^2 \rightarrow s^1 \Rightarrow s^2 = 0$

in BB framing, $s^2 = 3$

$0$-framing.

**Aside:** (Framed immersions $S^1 \rightarrow \mathbb{R}^3 \iff SO(3)$

\[
\begin{array}{c}
1 \\
\partial \rightarrow 1 \\
\partial \rightarrow 1
\end{array}
\]

Overall, $s^2$ is an integer, and as a function of $y$ alone, it is defined mod $\mathbb{Z}$.

\[
\eta(y) - s^2(y, y) = \lim_{t \to \infty} \int_{S^1} \text{ev}_t^* W
\]

\[
= \int_{S^1} (\text{a local quantity} \lambda \text{ compatible from } y, y \text{ near } \infty.)
\]

1. If $\gamma = \frac{y}{||y||}$ "the normal of $y$" then $\lambda$ is the Frénet–Serret torsion $\tau$ with $n = \frac{y}{||y||}$, $\tau = \dot{n} \cdot (\dot{t} \times n)$

2. Otherwise, $\lambda$ is "the drift rel. to the Riemannian connection".

3. An alternative definition of $s^2$:

$a \gamma$ defines a homotopy $H_t \gamma \rightarrow -\gamma$.
There is a pairing \( \langle u, \beta \rangle \in \mathbb{Z} \) between framings and swaddling maps:

\[
\phi^* \quad \stackrel{\Phi}{\rightarrow} \quad \mathbb{S}^2
\]

Overall get \( \Phi : T^2 \rightarrow \mathbb{S}^2 \)

Set \( s_2 = \deg \Phi \)

Then by declaring \( \beta \leftrightarrow u \iff \langle u, \beta \rangle = 0 \),
there is a bijection between (homotopy classes of) swaddling maps and odd framings.

If \( \beta \leftrightarrow u \), then \( s_2(u, \beta) = s_2(\Phi, \beta) \).

Proof: HW.