The linking number as a degree: Given \( \Phi : M \to N^m \) between oriented manifolds,

\[
\deg \Phi = \frac{\Phi^* [M^3]}{\Phi^* [W]} = \int_M \Phi^* \omega = \sum_{x \in \Phi^{-1}(y)} \deg_x \Phi
\]

is a homotopy invariant. \( l(y, \Phi) = \deg \Phi^* \omega \)

(The emptiest discussion in Mathematics - in two messy ways we will construct a knot invariant in \( \mathbb{Z}/2 \))

\[
\eta(y) = \frac{1}{4\pi} \int_N \Phi^* \omega = \frac{1}{4\pi} \oint_{S_1} \Phi^* \omega = 1
\]

\[
C_2(s') = \{(x, y) : \frac{x+1}{x+y} \leq 1, x \neq y, x = x+1 \}
\]

\[
\Phi_s(x, 0) = \lim_{t \to 0} \frac{\gamma(s+t, x) - \gamma_s(x)}{t} = \frac{\delta(x)}{1}
\]

\[
\Phi_s(x, 1) = \ldots = - \delta(x)
\]

\[
0 = \int_{C_2(S^1)} \Phi^* \omega = \int_{\partial C_2(S^1) \times I} \Phi^* \omega = \eta_1 - \eta_2 + 2 \int_{S^1} \Phi^* \omega
\]

Choose a "swallowing map" \( \beta : D^2 \to S^2 \) s.t. \( \beta|_{S^1} = \gamma \) and \( \beta^+ \to S^2 \)

\[
\text{def } s_l(\beta) = \deg \Phi
\]

\[
= \sum_x \Phi^* \omega \in \mathbb{Z}
\]

\* \( s_l(\emptyset, \emptyset) = 1 \)

\* In general, \( s_l(\beta) \) is an odd integer.

\* As a function of \( \gamma \) alone, it is defined up an even integer.

\[
\eta(y) = \frac{1}{4\pi} \int_N \Phi^* \omega
\]
Instead consider $d_2(x) = \ell(y, x^*(y))$

$y^*(x) = y(x) + x(y)$ ... framing, framed knots

Framings ∼ $\mathbb{Z}$ as an affine set

(compact two framings by a map $S^1 \to S^3$)

\[
\begin{align*}
\bigcirc & \quad d_2 = 0 \\
\bigcirc \otimes & \quad \text{o-framing, } d_2 = 3
\end{align*}
\]

Overall, $d_2$ is an integer, and as a function of $y$ alone, it is defined mod $\mathbb{Z}$.

\[
\eta(x) = \beta_2(x, y) = \int_\partial x \frac{\langle \omega, \lambda \rangle}{\beta_1(x, y)}
\]

\[
\left\{ \text{a bad quantity } \lambda \text{ compatible from } x, y \text{ near } x \right\}
\]

1. if $\lambda = \frac{\partial}{\partial x}$ "the normal of $y$" then

$\lambda$ is the Framet-Serret torsion $\tau$ with $\eta = \frac{\partial}{\partial x}$, $\tau = \eta \cdot (\delta \times \eta)$

2. Otherwise, $\lambda$ is "the drift rel. to the Riemannian connection".

Thus, there is a bijection between (homotopy classes of) swallow maps and framings.

If $x \leftrightarrow y$, then $d_2(x, y) = d_2(y, x)$