Friday-3 AKT on 140124: More Gaussian Integration

January-23-14 12:50 PM

HW session 6 PM ?

class photo @ 10:55 !

on board!

Goal for next 1-2 classes:

$$\int_{A\in\Omega^1(\mathbb{R}^3)} \mathcal{D}A\, e^{\frac{i}{4\pi}\int_{\mathbb{R}^3} A\wedge dA} \cdot \int_{\gamma_1} A \cdot \int_{\gamma_2} A = C\langle\gamma_1|d^{-1}|\gamma_2\rangle = C\cdot l(\gamma_1,\gamma_2)$$

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1. $$\int_{\mathbb{R}} e^{-\frac{\lambda}{2}x^2}dx = \sqrt{\frac{2\pi}{\lambda}}$$

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2. $$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j}dx = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x,\Lambda x\rangle}dx \qquad \Lambda=(\lambda_{ij}) \text{ positive definite}$$

$$= \frac{(2\pi)^{n/2}}{\det(\Lambda)^{1/2}} = C_\Lambda$$

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3. $$\int_{\mathbb{R}^n} P(x)e^{-\frac{1}{2}\langle x,\Lambda x\rangle}dx = P\left(\frac{\partial}{\partial y_1}\cdots\frac{\partial}{\partial y_n}\right)\int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x,\Lambda x\rangle+y\cdot x}dx\Big|_{y=0}$$

$$=P\left(\frac{\partial}{\partial y}\right)\int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle x-\Lambda^{-1}y,\Lambda(x-\Lambda^{-1}y)\rangle+\frac{1}{2}\langle y,\Lambda^{-1}y\rangle}dx\Big|_{y=0}$$

$$=\frac{(2\pi)^{n/2}}{\det(\Lambda)^{1/2}}P\left(\frac{\partial}{\partial y}\right)e^{\frac{1}{2}y\cdot\Lambda^{-1}y}\Big|_{y=0}$$

<u>Example</u> $$\int x^ix^je^{-\frac{1}{2}\lambda_{ij}x^ix^j}dx = \frac{(2\pi)^{n/2}}{(\det\Lambda)^{1/2}}\cdot\lambda^{ij}$$

Better yet, if $\varphi_1,\varphi_2\in V^*$, $\Lambda: V\to V^*$ is symmetric ($\Lambda^*=\Lambda$) and positive definite ($v\cdot\Lambda v=0 \Leftrightarrow v=0$)

and $$Z:=\int_{x\in V}dx\, e^{-\frac{1}{2}v\cdot\Lambda v},$$

Then $$Z^{-1}\int_{x\in V}dx\, e^{-\frac{1}{2}v\cdot\Lambda v}\varphi_1(v)\varphi_2(v) = \varphi_1\Lambda^{-1}\varphi_2$$

$$\Longrightarrow \quad Z(\gamma_1,\gamma_2) := Z^{-1}\int_{A\in\Omega^1(\mathbb{R}^3)} \mathcal{D}A\, e^{\frac{i}{4\pi}\int_{\mathbb{R}^3} A\wedge dA} \cdot \int_{\gamma_1} A \cdot \int_{\gamma_2} A = C\langle\gamma_1|d^{-1}|\gamma_2\rangle$$

done fine

More precisely, $\Lambda : \{1\text{-Forms}\} \longrightarrow \langle\text{Currents}\rangle$ $\quad j$

$\{1\text{-Forms}\} \xrightarrow{d} \{2\text{-forms}\}$, $\quad \langle\text{Currents}\rangle = \{2\text{-forms}\}$ $\quad \int_D \alpha = \int_D j\cdot\vec{n}$, $\quad \alpha$

$$\Lambda^{-1} \sim d^{-1} : \langle\text{Currents}\rangle \to \Omega^1$$

$j \longrightarrow$ the 1-form λ s.t.

$$\int_{\partial D} \lambda = \int_D d\lambda = \int_D j\cdot\vec{n}$$

$\Lambda^{-1}\gamma_2$: The 1-form λ whose integral around a small loop ∂D is 1 iff γ_2 pierces D positively.

$$\langle\gamma_1|\Lambda^{-1}|\gamma_2\rangle = \ell(\gamma_1,\gamma_2) \quad !$$

<u>Claim</u> If $\alpha\in\Omega^2(\mathbb{R}^3)$ & $d\alpha=0$, consider

$\phi : \mathbb{R}^3_x\times\mathbb{R}^3_y \longrightarrow S^2 \quad (x,y)\longmapsto \frac{x-y}{\|\ \ \|}$ $\quad w$ vol. on S^2

$\pi_{x,y} : \mathbb{R}^3_x\times\mathbb{R}^3_y \longrightarrow \mathbb{R}^3_x, \mathbb{R}^3_y$ projections

and set

$$\sigma = \int_{\mathbb{R}^3_y} \left(\phi^*(w)\wedge\pi_y^*\alpha\right) \in \Omega^1(\mathbb{R}^3_x)$$

Then $d\sigma = \alpha$

"a formula for d^{-1}"