

Gaussian Integration, Determinants, Feynman Diagrams

Gaussian Integration. (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of \mathbb{R}^n , let $(t_i)_{i=1}^n$ be a set of “dual” variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\begin{aligned}
 \int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{\epsilon}{6}\lambda_{ijk}x^i x^j x^k} &= \sum_{m \geq 0} \frac{\epsilon^m}{6^m m!} \int_{\mathbb{R}^n} (\lambda_{ijk} x^i x^j x^k)^m e^{-\frac{1}{2}\lambda_{ij}x^i x^j} \\
 &= \sum_{m \geq 0} \frac{C \epsilon^m}{6^m m!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m e^{\frac{1}{2}\lambda^{\alpha\beta} t_\alpha t_\beta} \Big|_{t_\alpha=0} = \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} (\lambda_{ijk} \partial^i \partial^j \partial^k)^m (\lambda^{\alpha\beta} t_\alpha t_\beta)^l \\
 &= \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} \left[\begin{array}{c} \text{... sum over all pairings ...} \\ \text{Diagrammatic expansion with vertices } \lambda_{i_1 j_1 k_1}, \dots, \lambda_{i_m j_m k_m} \end{array} \right] \\
 &= \sum_{\substack{m, l \geq 0 \\ 3m=2l}} \frac{C \epsilon^m}{6^m m! 2^l l!} \sum_{\text{m-vertex fully marked Feynman diagrams } D} \mathcal{E}(D) \\
 &= C \sum_{\text{unmarked Feynman diagrams } D} \frac{\epsilon^{m(D)} \mathcal{E}(D)}{|\text{Aut}(D)|}.
 \end{aligned}$$



Proof of the Claim. The group $G_{m,l} := [(S_3)^m \times S_m] \times [(S_2)^l \times S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D , and the stabilizer of any given P is $\text{Aut}(D)$. □

Determinants. Now suppose Q and P_i ($1 \leq i \leq n$) are $d \times d$ matrices and Q is invertible. Then

$$\begin{aligned}
 |\mathcal{Q}|^{-1} I_{\epsilon, \lambda_{ij}, \lambda_{ijk}, Q, P_i} &= |\mathcal{Q}|^{-1} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{\epsilon}{6}\lambda_{ijk}x^i x^j x^k} \det(Q + \epsilon x^i P_i) \\
 &= \sum_{m, k \geq 0, \sigma \in S_k} \frac{C \epsilon^{m+k} (-)^\sigma}{6^m m! k!} \int_{\mathbb{R}^n} (\lambda_{ijk} x^i x^j x^k)^m \text{tr}(\sigma(x^i Q^{-1} P_i)^{\otimes k}) e^{-\frac{1}{2}\lambda_{ij}x^i x^j} \\
 &= \sum_{\text{fully marked Feynman diagrams}} \frac{C \epsilon^{m+k} (-)^\sigma}{6^m m! k!} \mathcal{E} \left(\begin{array}{c} \text{Diagram with vertices } \lambda_{i_1 j_1 k_1}, \dots, \lambda_{i_m j_m k_m} \\ \text{and } k \text{ purple loops } \sigma \in S_k \end{array} \right) \\
 &= \sum_{\text{Feynman diagrams}} C \epsilon^{m+k} (-)^k (-)^l \mathcal{E} \left(\begin{array}{c} \text{Diagram with } m \text{ vertices and } k \text{ purple loops} \end{array} \right),
 \end{aligned}$$

where l is the number of purple (“Fermion”) loops.

Ghosts. Or else, introduce “ghosts” \bar{c}_a and c^b , write $I_{\epsilon, \lambda_{ij}, \lambda_{ijk}, Q, P_i} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\frac{1}{2}\lambda_{ij}x^i x^j + \frac{\epsilon}{6}\lambda_{ijk}x^i x^j x^k + \bar{c}_a(Q_a^a + \epsilon x^i P_{ib}^a) c^b}$ and use “ordinary” perturbation theory.

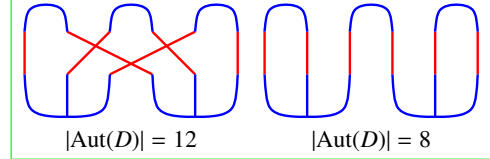


The Fourier Transform.

$(F: V \rightarrow \mathbb{C}) \Rightarrow (\tilde{f}: V^* \rightarrow \mathbb{C})$ via $\tilde{F}(\varphi) := \int_V f(v) e^{-i\langle \varphi, v \rangle} dv$. Some facts:

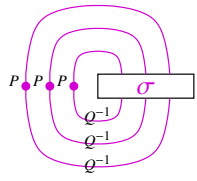
- $\tilde{f}(0) = \int_V f(v) dv$.
- $\frac{\partial}{\partial \varphi_i} \tilde{f} \sim v^i f$.
- $(e^{Q/2}) \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v \rangle$ for $L: V \rightarrow V^*$, and $Q^{-1}(\varphi) := \langle \varphi, L^{-1}\varphi \rangle$. (This is the key point in the proof of the Fourier inversion formula!)

Examples.



Perturbing Determinants. If Q and P are matrices and Q is invertible,

$$\begin{aligned}
 |\mathcal{Q}|^{-1} |\mathcal{Q} + \epsilon P| &= |I + \epsilon Q^{-1} P| \\
 &= \sum_{k \geq 0} \epsilon^k \text{tr} \left(\bigwedge^k Q^{-1} P \right) \\
 &= \sum_{k \geq 0, \sigma \in S_k} \frac{\epsilon^k (-)^\sigma}{k!} \text{tr}(\sigma(Q^{-1} P)^{\otimes k}) \\
 &= \sum_{k \geq 0, \sigma \in S_k} \frac{(-\epsilon)^k (-)^{\#\text{cycles}}}{k!} \text{tr}(\sigma(Q^{-1} P)^{\otimes k})
 \end{aligned}$$



The Berezin Integral (physics / math language, formulas from Wikipedia: Grassmann integral).

The Berezin Integral is linear on functions of anti-commuting variables, and satisfies $\int d\theta = 1$, and $\int 1 d\theta = 0$, so that $\int \frac{\partial f(\theta)}{\partial \theta} d\theta = 0$.



Let V be a vector space, $\theta \in V$, $d\theta \in V^*$ s.t. $\langle d\theta, \theta \rangle = 1$. Then $f \mapsto \int f d\theta$ is the interior multiplication map $\wedge V \rightarrow \wedge V$: $\int f d\theta := i_{d\theta}(f)$ ($= \frac{\partial f}{\partial \theta}$).

Multiple integration via “Fubini”: $\int f_1(\theta_1) \dots f_n(\theta_n) d\theta_1 \dots d\theta_n := (\int f_1 d\theta_1) \dots (\int f_n d\theta_n)$. $\int f d\theta_1 \dots d\theta_n := f // i_{d\theta_1} // \dots // i_{d\theta_n}$.

Change of variables. If $\theta_i = \theta_i(\xi_j)$, both θ_i and ξ_j are odd, and $J_{ij} := \partial \theta_i / \partial \xi_j$, then

$$\int f(\theta_i) d\theta = \int f(\theta_i(\xi_j)) \det(J_{ij})^{-1} d\xi.$$

Given vector spaces V_{θ_i} and W_{ξ_j} , $d\theta = \wedge d\theta_i \in \wedge^{\text{top}}(V^*)$, $d\xi = \wedge d\xi_j \in \wedge^{\text{top}}(W^*)$, and $T: V \rightarrow \wedge^{\text{odd}}(W)$. Then T induces a map $T_*: \wedge V \rightarrow \wedge W$ and then

$$\int f d\theta = \int (T_* f) \det \left(\frac{\partial(T\theta_i)}{\partial \xi_j} \right)^{-1} d\xi.$$

Gaussian integration. For an even matrix A and odd vectors θ, η , $\int e^{\theta^T A \eta} d\theta d\eta = \det(A)$, $\int e^{\theta^T A \eta + \theta^T J + \eta^T K} d\theta d\eta = \det(A) e^{-K^T A^{-1} J}$.