The Fulton-MacPherson Compactification

Let M be a manifold and let A be a finite set.

Definition 1. The open configuration space of A in M is $C^o_A(M) := \{\text{injections } p\colon A \to M\}$.

Definition 2. The compactified configuration space of A in M is

$$C_A(M) := \coprod_{\{A_1,\dots,A_k\},\ A=\cup A_\alpha,\ A_\alpha\neq\emptyset} \left\{\left(p_\alpha \in M, c_\alpha \in \tilde{C}_{A_\alpha}(T_{p_\alpha}M)\right)_{\alpha=1}^k : p_\alpha \neq p_\beta \text{ for } \alpha\neq\beta\right\}$$

where if V is a vector space and A is a singleton, $\tilde{C}_A(V) := \{\text{a point}\}$ and if $|A| \geq 2$,

$$\tilde{C}_A(V) := \coprod_{\substack{\{A_1,\dots,A_k\}\\ A=\cup A_\alpha;\ k\geq 2,\ A_\alpha\neq\emptyset}} \left\{\left(v_\alpha \in V, c_\alpha \in \tilde{C}_{A_\alpha}(T_{v_\alpha}V)\right)_{\alpha=1}^k : v_\alpha \neq v_\beta \text{ for } \alpha\neq\beta\right\} \Big/ \text{translations and dilations.}$$

Definition 3. A "d-manifold with corners" is defined in the same way as a manifold, except coordinate patches look like neighborhoods of 0 in $\mathbb{R}^d_{+k} := \{x \in \mathbb{R}^d\colon x^i \geq 0 \text{ for } i \leq k\}$ instead of merely like neigherborhoods of 0 in $\mathbb{R}^d$ or in $\mathbb{R}^d_+ := \{x \in \mathbb{R}^d\colon x^1 \geq 0\}$.

Theorem 4. $C_A(M)$ ia a manifold with corners, and $\partial C_A(M) = \coprod_{A'\subset A,\ |A'|\geq 2} \left\{(p,c)\colon\ p \in C^o_{A/A'}(M),\ c \in \tilde{C}_{A'}(T_{p_{A'}}M)\right\}$.

Theorem 5.
1. If M is compact, so is $C_A(M)$.
2. If A is a singleton, $C_A(M) = M$. If A is a doubleton, then $C_A(M)$ is isomorphic to $M\times M$ minus a tubular neighborhood of the diagonal $\Delta \subset M\times M$. That is, $C_A(M) = M\times M - V(\Delta)$.
3. If $B \subset A$ then there is a natural map $p_B\colon C_A(M) \to C_B(M)$. In particular, for every $i, j \in A$ there is a "direction map" $\phi_{ij}\colon C_A(\mathbb{R}^n) \to C_{\{i,j\}}(\mathbb{R}^n) \sim S^{n-1}$.
4. If $f: M \to N$ is a smooth embedding, then there's a natural $f_\star : C_A(M) \to C_A(N)$.

Now let D be a graph whose set of vertices is A. If two different vertices $a_{0,1} \in A$ are connected by an edge in D, we write $a_0 \overset{D}{\text{——}} a_1$. Likewise, if $A_{0,1} \subset A$ are disjoint subsets, we write $A_0 \overset{D}{\text{——}} A_1$ if $a_0 \overset{D}{\text{——}} a_1$ for some $a_0 \in A_0$ and $a_1 \in A_1$. For any subset A_0 of A we let $D(A_0)$ be the restriction of D to A_0.

Definition 6. The open configuration space of D in M is $C^o_D(M) := \{p\colon A \to M\colon\ p(a_0) \neq p(a_1) \text{ whenever } a_0 \overset{D}{\text{——}} a_1\}$.

Definition 7. The compactified configuration space of D in M is

$$C_D(M) := \coprod_{\substack{\{A_1,\dots,A_k\}\\ A=\cup A_\alpha,\ A_\alpha\neq\emptyset\\ \forall\alpha\ D(A_\alpha)\text{ connected}}} \left\{\left(p_\alpha \in M, c_\alpha \in \tilde{C}_{D(A_\alpha)}(T_{p_\alpha}M)\right)_{\alpha=1}^k : p_\alpha \neq p_\beta \text{ whenever } A_\alpha \overset{D}{\text{——}} A_\beta\right\}$$

where if V is a vector space and A is a singleton, $\tilde{C}_D(V) := \{\text{a point}\}$, and if $|A| \geq 2$,

$$\tilde{C}_D(V) := \coprod_{\substack{\{A_1,\dots,A_k\}\\ A=\cup A_\alpha;\ k\geq 2,\ A_\alpha\neq\emptyset\\ \forall\alpha\ D(A_\alpha)\text{ connected}}} \left\{\left(v_\alpha \in V, c_\alpha \in \tilde{C}_{D(A_\alpha)}(T_{v_\alpha}V)\right)_{\alpha=1}^k : v_\alpha \neq v_\beta \text{ whenever } A_\alpha \overset{D}{\text{——}} A_\beta\right\} \Big/ \text{translations and dilations.}$$

Theorem 8. The obvious parallel of the previous theorems holds.

Definition 9. Write $S^n = \mathbb{R}^n \cup \{\infty\}$ and set $\bar{C}_A(\mathbb{R}^n) := \{c \in C_{A\cup\{\infty\}}(S^n)\colon\ p_\infty(c) = \infty\}$.

Theorem 10. $\bar{C}_A(\mathbb{R}^n)$ is a compact manifold with corners and the direction maps $\phi_{ij}\colon \bar{C}_A(\mathbb{R}^n) \to S^{n-1}$ remain well-defined.

Finally, given $\gamma\colon S^1 \to \mathbb{R}^3$ and disjoint finite sets A and B, we set

$$C^\gamma_{A,B} := \left\{(c',c)\colon\ c' \in C_A(S^1),\ c \in \bar{C}_{A\cup B}(\mathbb{R}^3),\ \gamma_*(c') = p_A(c)\right\}$$

(and similarly C^γ_D for appropriate graphs D). The obvious variants of the theorems remain valid.