The Simplicity of the Alternating Groups

This handout is to be read twice: first read red only, to ascertain that everything in red is easy and boring, then read black and red, to actually understand the proof.

**Theorem.** The alternating group $A_n \leq S_n$ is simple for $n \neq 4$.

**Remark.** Easy for $n \leq 3$, false for $n=4$ as there is $\phi : A_4 \to A_3$, so assume $n \geq 5$.

**Lemma 1.** Every element of $A_n$ is a product of 3-cycles.

**Proof.** Every $\sigma \in A_n$ is a product of an even number of 2-cycles, and $(12)(23) = (123)$ & $(123)(23) = (12)(34)$.

**Lemma 2.** If $N \leq A_n$ contains a 3-cycle, then $N = A_n$.

**Proof.** WLOG, $(123) \in N$. Then for all $\sigma \in S_n$, $(123)^{-1} \in N$.

If $\sigma \in A_n$, this is clear. Otherwise $\sigma = (123)^{-1} \cdot (123)^{-1}$, and then as $(123)^{-1} = (123)^2$, $(123)^{-1} = (123)^2 \in N$. So $N$ contains all 3-cycles.

**Case 1.** $N$ contains an element with cycle of length $\geq 4$.

**Resolution.** $\sigma = (123456) \in N \implies \sigma^{-1}(123) \sigma^{-1} = (134) \in N$.

**Case 2.** $N$ contains an element with 2 cycles of length 3.

**Resolution.** $\sigma = (123)(456) \in N \implies \sigma^{-1}(124) \sigma^{-1} = (1243) \in N$.

**Case 3.** $N$ contains $\sigma = (123)$, a product of disjoint 2-cycles.

**Resolution.** $\sigma^{-2} = (132) \in N$.

**Case 4.** Every element of $N$ is a product of disjoint 2-cycles.

**Resolution.** $\sigma = (12)(34) \implies \sigma^{-1}(123) \sigma^{-1} = (13)(24) = \tau \in N \implies \tau^{-1}(125) \tau(125)^{-1} = (13452) \in N$. 

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