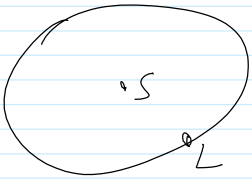
14-1100 Sep 25, hour 9: Group actions

September-17-14 2:07 PM

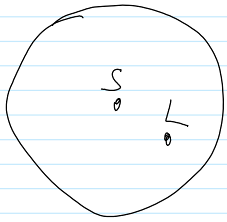
Class photo at end!

HW1 is out $\frac{1}{6}$

Riddle Along.



$V_L = 4V_S$



$V_S = V_L$

Today's Menu. Group actions.

Reminder. $G \circlearrowright X$, $X \circlearrowleft G$, both are categories! G/H

start line

Theorem. 1. Every G-set is a disjoint union of "transitive G-sets"

2. If X is a transitive G set and $x \in X$, then $X \cong G/\mathrm{stab}_X(x)$. (So $|X| \mid |G|$)

Theorem. If X is a G set and x_i are representatives of the orbits, then

$$|X| = \sum_i \frac{|G|}{|\mathrm{Stab}_X(x_i)|}$$

The class equation:

$$|G| = \underbrace{|Z(G)|}_{\text{the centre of } G} + \sum_i \underbrace{(G : C_G(y_i))}_{C_G(y_i) = \text{the centralizer of } y_i \text{ in } G}$$

Where $\{y_i\}$ are representatives from the non-central conjugacy classes of G.

Example. If G is a p-group, the centre of G is more than $\{e\}$.

done line

is more than $\{e\}$.

done
fin

THE SYLOW THEOREMS.

Lovely notation: $p^\alpha \| |G|$

$|G| = p^\alpha m$, p prime, $p \nmid m$; $Syl_p(G) := \{P < G : |P| = p^\alpha\}$ are "Sylow p-subgroups of G". A "p-subgroup" in general, is any subgroup of G of order a power of p.

Sylow 1 $Syl_p(G) \neq \emptyset$.

Proof. By induction on $|G|$, if G has a normal subgroup of order p (or p^β) or if G has a subgroup of order divisible by p^α, we are done. The existance of one of the said types follows from the class equation:

$$|G| = |Z(G)| + \sum_i (G : C_G(y_i))$$

Either both are divisible by p, or neither. Do 2^{nd} case first.

Where $\{y_i\}$ are representatives from the non-central conjugacy classes of G. $\square$

Theorem. If G is a finite Abelian group of order divisible by a prime p, then G contains an element of order p. "Cauchy's Thm" D&F pp 102

Proof. Enough to find an element of order divisible by p; if z is of order $p \cdot n$, z^n would be of order p. Pick $x \in G$, $x \neq 1$. If $p \mid |x|$, we're done. Otherwise $p \mid |G/\langle x \rangle|$, so by induction, $\exists y \in G$ s.t.

$|\bar{y}| = p$ in $G/\langle x \rangle$. Now use the following claim. $\square$

claim. if $\phi : G \to H$ is a morphism & $y \in G$, then $|\phi(y)| \,\big|\, |y|$.

Proof. If $|\phi(y)| = n$, $|y| = m$, $m = nq + r$, then

$$e = \phi(y^m) = \phi(y^{nq})\phi(y^r) = ((\phi(y))^n)^q \phi(y)^r = \phi(y)^r$$

So $r = 0$.

Stronger Sylow 1. If $p^\beta \,\big|\, |G|$, then G has a subgroup of order p^β.

Proof. Let $X = \{S \underset{\text{subset}}{\subset} G : |S| = p^\beta\}$, and write

$|G| = p^{\alpha+\beta} m$ w/ maximal α. By counting & binomial nonsense, $p^\alpha \,\big|\, |X|$ yet $p^{\alpha+1} \nmid |X|$. G acts on X by translations, so there must be $S_0 \in X$ s.t. $p^{\alpha+1} \nmid |G \cdot S_0|$, hence $p^\beta \,\big|\, |H = \mathrm{stab}_G(S_0)|$. Yet if $x \in S_0$ then $g \mapsto gx$ is an injection $H \to S_0$, so $|H| \leq |S_0| = p^\beta$, so $|H| = p^\beta$.