14-1100 Sep 22, hours 7-8: Simplicity of A_n, group actions

September-17-14 2:07 PM

HW1 is out $\ddot{\smile}$

Riddle Along. 1. Can you find uncountably many nearly-disjoint $[\forall \alpha,\beta\ |A_\alpha \cap A_\beta| < \infty]$ subsets of $\mathbb{N}$?

2. Can you find an uncountable chain $[\forall \alpha,\beta,\ (A_\alpha \subset A_\beta) \vee (A_\beta \subset A_\alpha)]$ of subsets of $\mathbb{N}$?

Today's Menu. Simplicity of A_n, Group actions.

Reminder. $\text{sign}: S_n \to \{\pm 1\}$ by $\text{sign}(\sigma) = (-1)^\sigma = \text{sign}\left(\prod_{i<j} (\sigma i - \sigma j)\right)$

$$= \prod_{\{i,j\} \subset \{1,\dots,n\}} s_{i,j}(\sigma) \qquad s_{i,j}(\sigma) = \text{sign}\left(\frac{\sigma i - \sigma j}{i-j}\right)$$

$$(-1)^{\sigma\tau} \overset{?}{=} \text{sign}\left[\prod_{\{i,j\}} \frac{\sigma\tau i - \sigma\tau j}{\tau i - \tau j} \cdot \frac{\tau i - \tau j}{i-j}\right] = (-1)^\sigma (-1)^\tau$$

Every permutation is a product of transpositions, the parity is the parity of the number of transpositions.

Theorem. A_n is simple for $n \neq 4$.

Cycle Decomposition. $(12)(345) = [21453] = 21453$

Claim If $\sigma = (a_1 \dots a_k)$ and $\tau = [\tau_1 \tau_2 \dots \tau_n]$, then

$$\sigma^\tau = \tau^{-1}\sigma\tau = (\tau^{-1}a_1, \tau^{-1}a_2, \dots)$$

Corollary σ is conjugate to σ' iff they have the same cycle lengths

Corollary $\#(\text{Conjugacy classes of } S_n) = p(n)$

Now follow handout....

Jordan-Hölder for S_n: $S_n \rhd A_n \rhd \{e\}$ $(n \geq 5)$

Definition A G-set (left-G-set) $G \times X \to X$

s.t. $(g_1 g_2)x = g_1(g_2 x)$, $ex = x$. Same as $\alpha: G \to S(X)$.

G-sets are a category!

Examples. 0. G itself, under mult. on the left.

1. G itself, under conjugation.

2. Subgroups(G), under conjugation.

Examples: 1. G/H when H is not-necessarily normal

Sub-example: S_n/S_{n-1} $\sigma S_{n-1} = \sigma' S_{n-1}$ iff $\sigma(n) = \sigma'(n)$. Let $\tau_i(n) = i$, then $\sigma\tau_i S_{n-1} = \tau_{\sigma i} S_{n-1}$. So S_n/S_{n-1} is $\{1 \ldots n\}$...

2. If X_1, X_2 are G-sets, then so is $X_1 \sqcup X_2$.

3. $S^2 = SO(3)/SO(2)$

done line

Theorem. 1. Every G-set is a disjoint union of "transitive G-sets"

2. If X is a transitive G set and $x \in X$, then $X \cong G/\mathrm{stab}_X(x)$. (So $|X| \mid |G|$)

Theorem. If X is a G set and x_i are representatives of the orbits, then

$$|X| = \sum_i \frac{|G|}{|\mathrm{Stab}_X(x_i)|}$$

Example. If G is a p-group, the centre of G is more than $\{e\}$.